K-GROUPS OF RECIPROCITY FUNCTORS

FLORIAN IVORRA AND KAY RÜLLING

Abstract. In this work we introduce reciprocity functors, construct the associated K-group functor of a family of reciprocity functors, which itself is a reciprocity functor, and compute it in several different cases. It may be seen as a first attempt to get close to the notion of reciprocity sheaves imagined by B. Kahn. Commutative algebraic groups, homotopy invariant Nisnevich sheaves with transfers, cycle modules or Kähler differentials are examples of reciprocity functors. As commutative algebraic groups do, reciprocity functors are equipped with symbols and satisfy a reciprocity law for curves.

Contents

The first author acknowledges support from the DAAD (Deutscher Akademischer Austausch Dienst) during the preparation of this work and thanks M. Levine for providing an excellent working environment and making his stay at the University Duisburg-Essen possible. He thanks B. Kahn for a discussion that arouses his interest for this topic. The second author was supported by the SFB/TR 45 “Periods, moduli spaces and arithmetic of algebraic varieties” of the DFG and thanks the first author for an invitation to the University of Rennes in 2010.

Introduction

Let F be a perfect field. In this work we introduce reciprocity functors. As cycle modules introduced by M. Rost in [27], reciprocity functors are functors with transfers and additional structures, defined on finitely generated field extensions of F. However as cycle modules are modeled after Milnor's K-theory, our model is the treatment of local symbols for smooth connected and commutative algebraic groups over an algebraically closed field as done by J.-P. Serre in [31]. These algebraic groups are not the only examples of reciprocity functors: homotopy invariant Nisnevich sheaves with transfers, cycle modules, Kähler differentials, all provide other examples of reciprocity functors.

Given reciprocity functors $\mathscr{M}_1, \ldots, \mathscr{M}_n$, our main construction is a reciprocity functor $\mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n)$ that we call the K-group of $\mathscr{M}_1, \ldots, \mathscr{M}_n$, although it is much more than a group, it is a reciprocity functor. This construction, though formally different, is related to the K-group attached by M. Somekawa in [32] to a family of semi-Abelian varieties and its variants introduced in [26, 2, 11]. We compute this K-group in several cases. This may be compared to results by W. Raskind-M. Spieß[26], R. Akhtar [2], B. Kahn-T. Yamazaki [16] and T. Hiranouchi [11]. This K-group is also related to the non-homotopy invariant world e.g. the additive Chow groups of S. Bloch and H. Esnault.

Detailed description of this work

Let F be a perfect field and $S = \operatorname{Spec} F$. A point over S or - for short - an S-point is the spectrum of a finitely generated field extension and we denote by $\mathrm{Reg}^{\leqslant 1}$ the category with objects the regular S-schemes of dimension $\leqslant 1$, which are separated and of finite type over some S-point. For example any curve which is regular and projective over some S-point is in $\mathrm{Reg}^{\leqslant 1}$ as well as all its points (closed or not) and non-empty open subsets. As in [35] we can define the category $\mathrm{Reg}^{\leqslant 1}\mathrm{Cor}$, which has the same objects as $\mathrm{Reg}^{\leqslant 1}$ but finite correspondences as morphisms.

Inspired by J.-P. Serre's treatment of local symbols for smooth connected and commutative algebraic groups over an algebraically closed field in [31] we define a *reciprocity functor* to be a presheaf $\mathscr{M}$ of Abelian groups on $\mathrm{Reg}^{\leqslant 1}\mathrm{Cor}$, which satisfies the following conditions:

(Nis) $\mathscr{M}$ is a sheaf in the Nisnevich topology on $\mathrm{Reg}^{\leqslant 1}$.

(F.P.) For all connected $X \in \mathrm{Reg}^{\leqslant 1}$ with generic point η, the group $\mathscr{M}(\eta)$ is the stalk in the generic point of $\mathscr{M}$ viewed as a Zariski sheaf on X.

(Inj) For all connected $X \in \mathrm{Reg}^{\leqslant 1}$ and all non-empty open subsets $U \subset X$ the restriction map $\mathscr{M}(X) \hookrightarrow \mathscr{M}(U)$ is injective.

(MC) For all regular projective and connected curves $C \in \mathrm{Reg}^{\leqslant 1}$, all non-empty open subsets $U \subset C$ and sections $a \in \mathscr{M}(U)$ there exists an effective divisor $\mathfrak{m}$ with support equal to $C \setminus U$ and

$$\sum_{P \in U} v_P(f) \mathrm{Tr}_{P/x_C} s_P(a) = 0,$$

where f is any non-zero element in the function field of C, which is congruent to 1 modulo $\mathfrak{m}$ (i.e. $\operatorname{div}(f-1) \geqslant \mathfrak{m}$), v_P is the discrete valuation associated to the closed point $P \in C$, $s_P : \mathscr{M}(U) \to \mathscr{M}(P)$ is the specialization map, which is simply given by the pullback along the natural

inclusion $P \hookrightarrow U$, $x_C = \mathrm{Spec}\ H^0(C, \mathscr{O}_C)$ and $\mathrm{Tr}_{P/x_C} : \mathscr{M}(P) \to \mathscr{M}(x_C)$ is the pushforward along the finite map $P \to x_C$.

Though the point of view of sheaves with transfers is quite convenient, we may give a description of reciprocity functors reminiscent of cycle modules (see Proposition 1.3.6). It is indeed easy to see that a presheaf on $\mathrm{Reg}^{\leqslant 1}\mathrm{Cor}$, which satisfies (F.P.) and (Inj) can be interpreted as a collection of groups $\mathscr{M}(x)$ for each S-point x together with pullback (resp. pushforward) maps for each (resp. finite) map between S-points, which satisfy certain natural compatibilities (so far one can call these data a Mackey functor), plus for all regular projective and connected curves C with generic point η and for all non-empty open subsets $U \subset C$ a subgroup $\mathscr{M}(U) \subset \mathscr{M}(\eta_C)$ on which a specialization map $s_P : \mathscr{M}(U) \to \mathscr{M}(P)$ is defined for all $P \in U$, which satisfy certain compatibilities. The condition (MC) is nothing but the modulus condition from [31]. Finally the reason that we work with the Nisnevich topology is that we want the transfers structure not to be destroyed by the sheafification process and as V. Voevodsky showed in [35], the Nisnevich topology is the right topology for this.

Our main examples of reciprocity functors are (see §2)

- Smooth commutative algebraic groups over S (this essentially follows from a theorem of M. Rosenlicht).
- Rost's cycle modules.
- By going to the generic stalks each homotopy invariant Nisnevich sheaf with transfers (in the sense of [35]) $\mathscr{F}$ defines a reciprocity functor denoted by $\hat{\mathscr{F}}$.
- Absolute Kähler differentials of degree $n \geqslant 0$, $x \mapsto \Omega^n_{x/\mathbb{Z}}$.

As a particular case of the second or third point above we also obtain

- For any smooth projective variety X over S of pure dimension d and any $n \geqslant 0$ the functor on S-points $x \mapsto \mathrm{CH}_0(X_x, n)$ defines a reciprocity functor, where $\mathrm{CH}_0(X_x, n)$ denotes Bloch's higher Chow groups.

In the same way as J.-P. Serre, we can show that a reciprocity functor $\mathscr{M}$ has local symbols. More precisely, if C is a regular projective and connected curve over some S-point x with generic point η, then for all closed points $P \in C$ there exists a bilinear map

$$(-,-)_P : \mathscr{M}(\eta) \times \mathbb{G}_m(\eta) \to \mathscr{M}(x),$$

which is continuous, when $\mathscr{M}(\eta)$ and $\mathscr{M}(x)$ are equipped with the discrete topology and $\mathbb{G}_m(\eta)$ with the $\mathfrak{m}_P$-adic toplogy ($\mathfrak{m}_P$ the maximal ideal in the local ring of C at P) and satisfies $(a, f)_P = v_P(f)\mathrm{Tr}_{P/x}(s_P(a))$, for all $a \in \mathscr{M}_{C,P}$, and the reciprocity law $\sum_{P \in C}(a, f)_P = 0$.

These local symbols provide an increasing and exhaustive filtration $\mathrm{Fil}^{\bullet}_P\mathscr{M}(\eta)$, where $\mathrm{Fil}^0_P\mathscr{M}(\eta) = \mathscr{M}_{C,P}$ and $\mathrm{Fil}^n_P\mathscr{M}(\eta)$ is the subgroup consisting of the elements $a \in \mathscr{M}(\eta)$ such that $(a, 1+\mathfrak{m}_P^n)_P = 0$. We denote by $\mathbf{RF}$ the category of reciprocity functors and by $\mathbf{RF}_n$ the full subcategory consisting of those reciprocity functors such that $\mathrm{Fil}^n_P\mathscr{M}(\eta) = \mathscr{M}(\eta)$ for all $P \in C$ as above. Then Abelian varieties or $\mathrm{CH}_0(X)$, with X smooth projective, lie in $\mathbf{RF}_0$, semi-Abelian varieties, Rost's cycle modules and homotopy invariant Nisnevich sheaves with transfers lie in $\mathbf{RF}_1$, whereas $\mathbb{G}_a$ or more general the Kähler differentials do not lie in any $\mathbf{RF}_n$ (since the pole order is not bounded). Notice that for the homotopy invariant examples above there in fact exists a *fine* symbol with values in the residue fields; it seems that in general one cannot hope to get something like this, e.g. in the case of the additive group $\mathbb{G}_a$ the best the authors can get for $P \in C$ as above is a symbol with values in in the separable closure of x in P.

Main results

Now let $\mathscr{M}_1, \dots, \mathscr{M}_n$ and $\mathscr{N}$ be reciprocity functors. Then an n-linear map of reciprocity functors $\Phi : \mathscr{M}_1 \times \ldots \times \mathscr{M}_n \to \mathscr{N}$ is an n-linear map of sheaves, which is compatible with pullback, satisfies a projection formula and the following condition

$$\text{(L3)} \quad \Phi(\mathrm{Fil}_P^{r_1} \mathscr{M}_1(\eta) \times \ldots \mathrm{Fil}_P^{r_n} \mathscr{M}_n(\eta)) \subset \mathrm{Fil}_P^{\max\{r_1,\dots,r_n\}},$$

for all regular projective curves C with generic point η and $P \in C$ a closed point and all positive integers $r_1, \dots, r_n \geqslant 1$. We denote by $n - \mathrm{Lin}(\mathscr{M}_1, \dots, \mathscr{M}_n; \mathscr{N})$ the group of n-linear maps as above. Then the main theorem of this article is the following:

Theorem (see Theorem 4.2.4)**.** *The functor* $\mathbf{RF} \to$ *(Abelian groups),* $\mathscr{N} \mapsto n - \mathrm{Lin}(\mathscr{M}_1, \dots, \mathscr{M}_n; \mathscr{N})$ *is representable by a reciprocity functor*

$$\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n).$$

We call $\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n)$ the K-group of $\mathscr{M}_1, \dots, \mathscr{M}_n$, although it is much more than a group, it is a reciprocity functor. We would like to call this a tensor product, unfortunately it is not clear whether associativity is satisfied (one reason is the condition (L3)). But all other properties of a tensor product hold: we have commutativity, compatibility with direct sums, the constant reciprocity functor $\mathbb{Z}$ is a unit object and it is right exact (the category of reciprocity functors is quasi-Abelian in the sense of [29], in particular we have kernels and cokernels and a notion of short exact sequences and right exact functors on it, see §3).

We have the following computations:

Theorem (see Theorem 5.1.8)**.** *Let* $\mathscr{F}_1, \dots, \mathscr{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$ *be homotopy invariant Nisnevich sheaves with transfers. There exists a canonical and functorial isomorphism of reciprocity functors*

$$\mathrm{T}(\hat{\mathscr{F}}_1, \dots, \hat{\mathscr{F}}_n) \xrightarrow{\sim} (\mathscr{F}_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathscr{F}_n)^{\widehat{}}.$$

This theorem should be compared to the main theorem of [16], where B. Kahn and T. Yamazaki prove that the right hand side evaluated at S is isomorphic to $\mathrm{K}(S, \mathscr{F}_1, \dots, \mathscr{F}_n)$, a variant of Somekawa's K-groups. In particular we obtain that over a perfect field $\mathrm{T}(G_1, \dots, G_n)(S)$ with G_i semi-Abelian varieties equals Somekawa's K-group at S. In the same way as in [16] we obtain as a corollary:

Corollary (see Corollary 5.2.5)**.** *Let* $X_1, \dots, X_n$ *be smooth projective schemes over* S *and* $r \geqslant 0$ *an integer. Then for all* S*-points* x *we have an isomorphism*

$$\mathrm{T}(\mathrm{CH}_0(X_1), \dots, \mathrm{CH}_0(X_n), \mathbb{G}_m^{\times r})(x) \cong \mathrm{CH}_0(X_{1,x} \times_x \ldots \times_x X_{n,x}, r).$$

Using a variant of Somekawa's K-groups on the left hand side the above isomorphism was proven by W. Raskind and M. Spieß in [26] (case $r = 0$) and by R. Akhtar in [2] (case $r \geqslant 0$).

We can use this as in [32] or [26] to determine the kernel of the Albanese map of a product of smooth projective and connected curves over $S = \operatorname{Spec} F$, where F is algebraically closed. If e.g. C_1, C_2 are two such curves then the choice of an F-rational point in C_i yields a splitting of $\mathrm{CH}_0(C_i)$ as a direct sum of reciprocity functors $\mathrm{CH}_0(C_i)^0 \oplus \mathbb{Z}$, where $\mathrm{CH}_0(C_i)^0$ is the group of zero cycles of degree 0 on C_i (it is a reciprocity functor since it is the kernel of the degree map, which is a map of reciprocity functors). Further denote J_i the Jacobian of C_i. Then

$$\begin{aligned} \mathrm{CH}_0(C_1 \times C_2) &= \mathrm{T}\left(\mathrm{CH}_0(C_1)^0 \oplus \mathbb{Z}, \mathrm{CH}_0(C_2)^0 \oplus \mathbb{Z}\right)(S) \\ &= \mathrm{T}\left(\mathrm{CH}_0(C_1)^0, \mathrm{CH}_0(C_2)^0\right)(S) \oplus J_1(S) \oplus J_2(S) \oplus \mathbb{Z}. \end{aligned}$$

Thus $\mathrm{T}\left(\mathrm{CH}_0(C_1)^0, \mathrm{CH}_0(C_2)^0\right)(S)$ is the kernel of the Albanese map

$$\mathrm{alb}: \mathrm{CH}_0(C_1 \times C_2)^0 \to \mathrm{Alb}(C_1 \times C_2)(S) = J_1(S) \oplus J_2(S).$$

Further we can also calculate by hand:

Theorem (see Theorem 5.3.3). *For all $n \geqslant 1$ and all S-points $x = \mathrm{Spec}\, k$, there is canonical isomorphism*

$$\mathrm{T}(\mathbb{G}_m^{\times n})(x) \xrightarrow{\simeq} \mathrm{K}_n^{\mathrm{M}}(k),$$

where $\mathrm{K}_n^{\mathrm{M}}(k)$ denotes the n-th Milnor K-group of k.

Observe that combining the corollary above in the case $n = 1$ and $X_1 = S$ with this theorem we get the Nesterenko-Suslin theorem $\mathrm{CH}^n(k, n) \cong \mathrm{K}_n^{\mathrm{M}}(k)$. But actually to prove the above theorem we use more or less the same methods as Yu. Nesterenko and A. Suslin, so it is not really a new proof of their theorem.

Theorem (see Theorem 5.4.7). *Assume F has characteristic zero. Then there is an isomorphism for all S-points x*

$$\theta : \Omega^n_{x/\mathbb{Z}} \xrightarrow{\simeq} \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x).$$

Combining the above theorem with the theorem of S. Bloch and H. Esnault in [4] which identifies $\Omega^{n-1}_{x/Z}$ with the additive higher Chow groups of x of level n (and with modulus 2) , we obtain

$$\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n-1})(x) \cong \mathrm{TCH}^n(x, n, 2).$$

This result does not hold in positive characteristic. The reason is essentially that in positive characteristic the algebraic group (or the reciprocity functor) $\mathbb{G}_a$ has more endomorphisms than in characteristic zero, namely the absolute Frobenius comes into the game. This forces the following

Corollary (see Corollary 5.4.12). *Assume F has characteristic $p > 0$. Then for all S-points x we have a surjective morphism*

$$\Omega^n_{x/\mathbb{Z}}/B_\infty \twoheadrightarrow \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times})(x)$$

and the following commutative diagram

$$\begin{tikzcd}
\Omega^n_{x/\mathbb{Z}}/B_\infty \arrow[r, "C^{-1}"] \arrow[d] & \Omega^n_{x/\mathbb{Z}}/B_\infty \arrow[d] \\
\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x) \arrow[r, "F\otimes\mathrm{id}"] & \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times})(x),
\end{tikzcd}$$

where $F : \mathbb{G}_a \to \mathbb{G}_a$ is the absolute Frobenius, $C^{-1} : \Omega^n_{x/\mathbb{Z}} \to \Omega^n_{x/\mathbb{Z}}/d\Omega^{n-1}_{x/\mathbb{Z}}$ is the inverse Cartier operator and B_∞ is the union over B_n, where $B_1 = d\Omega^{n-1}_{x/\mathbb{Z}}$ and B_n is the preimage of $C^{-1}(B_{n-1})$ in $\Omega^n_{x/\mathbb{Z}}$ for $n \geqslant 2$.

We don't know whether the above map is an isomorphism. The above theorem and the corollary should be compared to [11], where T. Hiranouchi defines a variant of Somekawa's K-groups for perfect fields in which one can also plug in $\mathbb{G}_a$ (and also Witt group schemes) and in [11, Theorem 3.6] he proves $\Omega^n_{F/\mathbb{Z}} \cong \mathrm{K}(F; \mathbb{G}_a, \mathbb{G}_m^n)$. (Notice that in positive characteristic this is just a vanishing result for the K-group, which for our group also follows from the above corollary.)

Let us also mention the following vanishing result:

Theorem (see Theorem 5.5.1). *Assume* $\mathrm{char}(F) \neq 2$. *Let $\mathscr{M}_1, \ldots, \mathscr{M}_n$ be reciprocity functors and assume that at least* two *of them are smooth connected commutative* unipotent *group schemes over S. Then*

$$\mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n) = 0.$$

Finally, we want to stress, that we tried to find a minimal set of axioms for a reciprocity functor in order to obtain the above results. There are further very natural axioms which one could impose, e.g. that the composition of specialization maps is independent of the chosen specialization chain, see for example Morel's axiom (A4) in the definition of an unramified $\mathscr{F}_k$-datum in [24, Definition 1.9]. The authors hope that there might be some extra axioms to impose on a reciprocity functor, which could help to get around some difficulties, e.g. that we don't know whether **RF** is an abelian category (with the current definition it is probably not) or the missing associativity for the T-functor. Thus what we call reciprocity functor should maybe called a pre-reciprocity functor and the actual definition of a reciprocity functor is still to be given.

Conventions and notations

We fix the following notations and conventions:

(1) F is a perfect field and $S = \operatorname{Spec} F$ is our base point. A point x over S is a morphism $\operatorname{Spec} k \to S$, where k is a finitely generated field extension over F. By abuse of notation we will also say that $x = \operatorname{Spec} k$ is a point over S, meaning that it comes with a fixed morphism to S and (by even more abuse) we will simply say *x is an S-point*. If not said differently cartesian products of schemes are over S.
(2) R is a commutative ring with 1 and we denote by $(R-\mathrm{mod})$ the category of R-modules.
(3) By a curve over an S-point x we mean a pure 1-dimensional scheme, which is separated and of finite type over x. Points on a curve C over an S-point x, which are denoted by capital letters like $P, Q, P', \ldots \in C$ are always meant to be closed points; if C is irreducible its generic point will be denoted by η_C or just by η if no confusion can arise.
(4) If X is a scheme and $x \in X$ is a point, we denote by $\kappa(x)$ its residue field, if X is integral and η is its generic point then we also write $\kappa(X) = \kappa(\eta)$ for its function field.
(5) If $f : X \to Y$ is a morphism of S-schemes we denote by $\Gamma_f \subset X \times Y$ its graph and by Γ_f^t its transpose.
(6) If C is a regular connected curve over a field and $P \in C$ is a closed point, then we set for a positive integer n
$$U_P^{(n)} := 1 + \mathfrak{m}_P^n,$$
where $\mathfrak{m}_P \subset \mathscr{O}_{C,P}$ denotes the maximal ideal. Further we denote by $v_P : \kappa(C) \to \mathbb{Z} \cup \{\infty\}$ the normalized discrete valuation associated to P.
If $D \to C$ is a finite and surjective morphism between regular curves over a field, sending a closed point $Q \in D$ to $P \in C$, we will denote by $e(Q/P)$ and $f(Q/P) = [\kappa(Q) : \kappa(P)]$ the ramification index and the inertia degree, respectively.
(7) If X is an integral scheme we denote by $\tilde{X}$ the normalization of X and by $\nu_X : \tilde{X} \to X$ the corresponding map.

1. Reciprocity Functors

1.0.1. We introduce the following categories:

(1) Let (pt/S) be the category with objects the S-points x and morphisms the S-morphisms $x \to y$ of S-points.
(2) Let $(\mathrm{pt}/S)_*$ be the category with objects the S-points and morphisms the *finite* S-morphisms of S-points.

(3) Let $(\mathscr{C}/S)$ be the category with objects the regular, connected and projective curves C over some S-point, morphisms are dominant morphisms of S-schemes.
(4) Let $\mathrm{Reg}_S^{\leqslant 1} = \mathrm{Reg}^{\leqslant 1}$ be the category with objects the regular S-schemes of dimension $\leqslant 1$, which are separated and of finite type over some S-point, morphisms are morphisms of S-schemes.
(5) Let $\mathrm{RegCon}^{\leqslant 1}$ (resp. $\mathrm{RegCon}_*^{\leqslant 1}$) be the category with objects the connected regular S-schemes of dimension $\leqslant 1$ which are separated and of finite type over some S-point, morphisms are morphisms of S-schemes (resp. finite flat morphism of S-schemes).

Notice that (pt/S) is a subcategory of $\mathrm{Reg}^{\leqslant 1}$ and that we have a faithful functor $(\mathscr{C}/S) \to \mathrm{Reg}^{\leqslant 1}$. Further if U is in $\mathrm{Reg}^{\leqslant 1}$, then any point $y \in U$ naturally defines an S-point. If $U \in \mathrm{Reg}^{\leqslant 1}$ is connected and 1 dimensional, then it admits an open immersion into a curve $C \in (\mathscr{C}/S)$ and C is unique up to isomorphism. For a curve $C \in (\mathscr{C}/S)$, we denote

$$x_C := \operatorname{Spec} H^0(C, \mathscr{O}_C) \in (\mathrm{pt}/S), \tag{1.1}$$

Notice that $C \to x_C$ is projective and geometrically connected.

1.1. **Correspondences in dimension at most 1.**

1.1.1. **Definition.** Let X and Y be in $\mathrm{Reg}^{\leqslant 1}$. Then we denote by $\mathrm{Cor}_R(X, Y) = \mathrm{Cor}(X, Y)$ the free R-module generated by symbols $[V]$ with V an integral closed subscheme of $X \times Y$, which is finite and surjective over a connected component of X. Elements in $\mathrm{Cor}(X, Y)$ are called correspondences from X to Y and correspondences of the form $[V]$ with V as above are called elementary correspondences.

1.1.2. **Lemma.** *Let X, Y and Z be in $\mathrm{Reg}^{\leqslant 1}$ and let $[V] \in \mathrm{Cor}(X, Y)$, $[W] \in \mathrm{Cor}(Y, Z)$ be elementary correspondences. Let $(V \times Z) \cap (X \times W)$ be the intersection in $X \times Y \times Z$ and let T be an irreducible component with generic point η. Then T as well as its image in $X \times Z$ are finite and surjective over a connected component of X and*

$$\mathrm{Tor}_i^{\mathscr{O}_{X\times Y\times Z,\eta}}(\mathscr{O}_{V\times Z,\eta}, \mathscr{O}_{X\times W,\eta}) = 0 \quad \textit{for all } i \geqslant 1.$$

Proof. For the first statement see e.g. [22, 1.]. To prove the vanishing of the higher Tor's we can assume that the schemes V, W, X, Y, Z are all integral of dimension $\leqslant 1$ and hence are CM. We can spread them out to integral schemes $\bar{V}, \bar{W}, \bar{X}, \bar{Y}, \bar{Z}$, which are separated and of finite type over S such that $\bar{X}$, $\bar{Y}$, $\bar{Z}$ are smooth over S (this is possible since S is perfect) and $\bar{V} \subset \bar{X} \times \bar{Y}$ and $\bar{W} \subset \bar{Y} \times \bar{X}$ are closed subschemes, which are finite and surjective over the first factor and which we can assume to be CM (by [10, Cor. (6.4.2)] applied to $V \to \bar{V}$, $W \to \bar{W}$). Furthermore we can assume that T spreads out to an integral closed subscheme $\bar{T} \subset (\bar{V} \times \bar{Z}) \cap (\bar{X} \times \bar{W})$, which is finite and surjective over $\bar{X}$. Since it clearly suffices to prove the vanishing of the over-lined-Tor's we can assume in the following that V, W, X, Y, Z are integral, separated and of finite type over S (of some positive dimension) with X, Y, Z smooth over S and V, W CM. We set $A := \mathscr{O}_{X\times Y\times Z,\eta}$, $A_V := \mathscr{O}_{V\times Z,\eta}$, $A_W := \mathscr{O}_{X\times W,\eta}$. Since A is regular the kernel of

$$B := A \otimes_F A \to A, \quad a \otimes b \mapsto ab$$

is generated by a regular sequence $\underline{t} = t_1, \ldots, t_n$, with $n := \dim A$. Hence the Koszul complex $K_\bullet^B(\underline{t})$ is a free resolution of the B-module A. By [10, Cor. (6.7.3)] the B-algebra $A_V \otimes_F A_W$ is CM of dimension

$$\dim(A_V \otimes_F A_W) = \dim A_V + \dim A_W = \dim Y + \dim Z = \dim A = n$$

and

$$(A_V \otimes_F A_W)/(\underline{t}) = (A_V \otimes_F A_W) \otimes_B A = A_V \otimes_A A_W = \mathscr{O}_{V\times_Y W,\eta}$$

has finite length. Thus $\underline{t}$ is a system of parameters for the CM B-module $A_V \otimes_F A_W$ and we obtain for $i \geqslant 1$

$$\begin{aligned} \mathrm{Tor}_i^A(A_V, A_W) &\cong \mathrm{Tor}_i^B(A_V \otimes_F A_W, A), && \text{see e.g. [30, V,(2)],} \\ &= H_i(K_\bullet^B(\underline{t}) \otimes_B (A_V \otimes_F A_W)) \\ &= 0, && \text{by [30, IV, Thm 3, iv)].} \end{aligned}$$

Hence the statement. □

1.1.3. **Definition.** (1) Let X, Y and Z be in $\mathrm{Reg}^{\leqslant 1}$ and let $[V] \in \mathrm{Cor}(X, Y)$ and $[W] \in \mathrm{Cor}(Y, Z)$ be elementary correspondences. Then we define

$$[W] \circ [V] := \sum_{T \subset V\times_Y W} \mathrm{lg}(\mathscr{O}_{V\times_Y W,\eta_T}) \cdot p_{XZ*}[T],$$

where T runs over the irreducible components of $V \times_Y W$, η_T is the generic point of T and $p_{XZ} : V \times_Y W \to X \times Z$ is the natural map induced by projection. It follows from Lemma 1.1.2 that $[W] \circ [V]$ is an element in $\mathrm{Cor}(X, Z)$ and coincides with the pushforward to $X \times Z$ of the intersection cycle $[V \times Z] \cdot [X \times W]$ (in the sense of [30, V]). Hence $\circ$ extends to an R-bilinear pairing

$$\mathrm{Cor}(X, Y) \times \mathrm{Cor}(Y, Z) \to \mathrm{Cor}(X, Z), \quad (\alpha, \beta) \mapsto \beta \circ \alpha,$$

which is associative and with the diagonal $\Delta \subset Y \times Y$ acting from the right (resp. left) as identity on $\mathrm{Cor}(X, Y)$ (resp. $\mathrm{Cor}(Y, X)$).

(2) We define the category $\mathrm{Reg}^{\leqslant 1}\mathrm{Cor}$ to be the category with objects the objects of $\mathrm{Reg}^{\leqslant 1}$ and morphisms given by correspondences. We denote by ptCor the full subcategory with objects the finite disjoint unions of S-points.

Notice that $\mathrm{Reg}^{\leqslant 1}\mathrm{Cor}$ is an R-linear additive category and that the graph-construction gives in the usual way a faithful and essentially surjective functor $\mathrm{Reg}^{\leqslant 1} \to \mathrm{Reg}^{\leqslant 1}\mathrm{Cor}$.

1.1.4. **Lemma.** *(1) Let X and Y be in $\mathrm{Reg}^{\leqslant 1}$ and let V be an integral scheme with a map $\pi : V \to X \times Y$ such that V is finite and surjective over X. Let $\tilde V$ be the normalization of V and $q_V : \tilde V \to X$ and $p_V : \tilde V \to Y$ be induced by the projection maps. Then $\tilde V \in \mathrm{Reg}^{\leqslant 1}$ and*

$$\pi_*[V] = [\Gamma_{p_V}] \circ [\Gamma^t_{q_V}] \in \mathrm{Cor}(X, Y).$$

(2) Let $f : X \to Y$ be a finite and surjective map in $\mathrm{RegCon}^{\leqslant 1}$. Then

$$[\Gamma_f] \circ [\Gamma_f^t] = \deg(X/Y) \cdot [\Delta_Y],$$

with $\Delta_Y \subset Y \times Y$ the diagonal.

(3) For any cartesian square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

where Y, X, X' are in $\mathrm{RegCon}^{\leqslant 1}$, $g : X' \to X$ is a morphism and $f : Y \to X$ is a finite flat morphism, we have

$$[\Gamma^t_f] \circ [\Gamma_g] = \sum_T \mathrm{lg}(\mathscr{O}_{Y',\eta_T}) \cdot [\Gamma_{g_T}] \circ [\Gamma^t_{f_T}] \quad \textit{in } \mathrm{Cor}(X', Y),$$

where the sum is taken over the irreducible components T of Y', and for such an irreducible component $g_T : \tilde{T} \to Y$ and $f_T : \tilde{T} \to X'$ are the canonical morphisms from the normalization $\tilde{T}$ of T.

Proof. (1) follows from $\Gamma^t_{q_V} \times_{\tilde{V}} \Gamma_{p_V} \cong \tilde{V}$, (2) from $\Gamma^t_f \times_X \Gamma_f \cong \Delta_X$ and (3) follows from $\Gamma_g \times_X \Gamma^t_f \cong X' \times_X Y \cong Y'$, the definition of the composition and (1). □

1.1.5. **Lemma.** *Let X, Y, Z be in $\mathrm{Reg}^{\leqslant 1}$ and $[V] \in \mathrm{Cor}(X, Y)$, $[W] \in \mathrm{Cor}(Y, Z)$ be elementary correspondences. Let T be an irreducible component of $V \times_Y W$. Then*

$$\lg(\mathscr{O}_{V\times_Y W,\eta_T}) = \sum_{C/T} \lg(\mathscr{O}_{\tilde{V}\times_Y \tilde{W},\eta_C}) \deg(C/T). \tag{1.2}$$

where the sum is taken over the irreducible components C of $\tilde{V} \times_Y \tilde{W}$ that dominate T.

Proof. Using Lemma 1.1.4, (1) and (3), we obtain:

$$\begin{aligned}
[W] \circ [V] &= [\Gamma_{p_W}] \circ [\Gamma^t_{q_W}] \circ [\Gamma_{p_V}] \circ [\Gamma^t_{q_V}] \\
&= [\Gamma_{p_W}] \circ \left(\sum_{C \subseteq \tilde{V}\times_Y \tilde{W}} \lg(\mathscr{O}_{\tilde{V}\times_Y \tilde{W},\eta_C}) \cdot [\Gamma_{p'_C}] \circ [\Gamma^t_{q'_C}] \right) \circ [\Gamma^t_{q_V}] \\
&= \sum_{C \subseteq \tilde{V}\times_Y \tilde{W}} \lg(\mathscr{O}_{\tilde{V}\times_Y \tilde{W},\eta_C}) \cdot [\Gamma_{p_C}] \circ [\Gamma^t_{q_C}],
\end{aligned}$$

where the sum is taken over the irreducible components C of $\tilde{V} \times_Y \tilde{W}$, the maps

$$X \xleftarrow{q_V} \tilde{V} \xrightarrow{p_V} Y\ , \quad Y \xleftarrow{q_W} \tilde{W} \xrightarrow{p_W} Z\ , \quad \tilde{V} \xleftarrow{q'_C} \tilde{C} \xrightarrow{p'_C} \tilde{W}$$

are the canonical ones and $p_C = p_W \circ p'_C$ and $q_C = q_V \circ q'_C$. Since $\tilde{V} \times_Y \tilde{W}$ is finite and surjective over $V \times_Y W$, any irreducible component C of $\tilde{V} \times_Y \tilde{W}$ maps surjectively onto an irreducible component T of $V \times_Y W$ and the induced morphism $\tau_C : \tilde{C} \to \tilde{T}$ is finite surjective and hence is flat. Further, denoting by $p_T : \tilde{T} \to Z$ and $q_T : \tilde{T} \to X$ the canonical maps we have $p_T \circ \tau_C = p_C$ and $q_T \circ \tau_C = q_C$. Therefore we obtain from Lemma 1.1.4, (2) and (1)

$$[W] \circ [V] = \sum_{T \subseteq V\times_Y W} \left(\sum_{C/T} \lg(\mathscr{O}_{\tilde{V}\times_Y \tilde{W},\eta_C}) \deg(\eta_C/\eta_T) \right) \cdot p_{XZ*}[T]$$

where the second sum is taken over the irreducible components C of $\tilde{V} \times_Y \tilde{W}$ that dominate T. Comparing this with the formula given in Definition 1.1.3 yields (1.2). □

1.2. **Presheaves with transfers.**

1.2.1. **Definition.** A *presheaf with transfers on* $\mathrm{Reg}^{\leqslant 1}$ is an R-linear contravariant functor

$$\mathscr{M} : \mathrm{Reg}^{\leqslant 1}\mathrm{Cor}^{\mathrm{op}} \to (R - \mathrm{mod}).$$

We denote by **PT** the category of presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$ with morphisms the natural transformations. Further we denote by **NT** the full subcategory with objects the *Nisnevich sheaves with transfers on* $\mathrm{Reg}^{\leqslant 1}$, i.e. those presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$ whose underlying presheaf on $\mathrm{Reg}^{\leqslant 1}$ is a sheaf in the Nisnevich topology.

1.2.2. **Lemma.** *A presheaf with transfers on* $\mathrm{Reg}^{\leqslant 1}$ *is equivalent to the following data*

- *a functor* $\mathscr{M}_* : \mathrm{RegCon}^{\leqslant 1}_* \to (R - \mathrm{Mod})$*;*

- *a functor* $\mathscr{M}^* : \mathrm{RegCon}^{\leqslant 1\,\mathrm{op}} \to (R-\mathrm{Mod})$*;*

which satisfies the following conditions (where we write $f_* := \mathscr{M}_*(f)$ *and* $f^* := \mathscr{M}^*(f)$ *for a map* f *in* $\mathrm{RegCon}^{\leqslant 1}_*$ *and* $\mathrm{RegCon}^{\leqslant 1}$*, respectively)*

(1) $\mathscr{M}_*(X) = \mathscr{M}^*(X)$ *for any* $X \in \mathrm{RegCon}^{\leqslant 1}$*;*
(2) $g_*g^* = \deg(Y/X)\cdot \mathrm{id}_{\mathscr{M}(X)}$ *for any finite flat morphism* $g : Y \to X$ *between connected schemes in* $\mathrm{Reg}^{\leqslant 1}$*;*
(3) *for any cartesian square*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

where Y, X, X' *are in* $\mathrm{RegCon}^{\leqslant 1}$*,* $g : X' \to X$ *is a morphism and* $f : Y \to X$ *is a finite flat morphism, we have*

$$g^* \circ f_* = \sum_T \mathrm{lg}(\mathscr{O}_{Y',\eta_T})\cdot f_{T*}\circ g_T^*,$$

where the sum is taken over the irreducible components T *of* Y'*, and for such an irreducible component* $g_T : \tilde{T} \to Y$ *and* $f_T : \tilde{T} \to X'$ *are the canonical morphisms.*

Proof. That a presheaf with transfers on $\mathrm{RegCon}^{\leqslant 1}$ defines such data is an immediate consequence of Lemma 1.1.4. Conversely let $\mathscr{M}_*$ and $\mathscr{M}^*$ be such a data satisfying (1)–(3). Set $\mathscr{M}(X) := \mathscr{M}_*(X) = \mathscr{M}^*(X)$ for any $X \in \mathrm{RegCon}^{\leqslant 1}$ and $\mathscr{M}(X) := \oplus_i \mathscr{M}(X_i)$ for $X \in \mathrm{Reg}^{\leqslant 1}$ with connected components X_i. For an elementary correspondence $[V] \in \mathrm{Cor}(X, Y)$ we define a morphism

$$\mathscr{M}([V]) := q_{V*}p_V^* : \mathscr{M}(Y) \to \mathscr{M}(X),$$

where $q_V : \tilde{V} \to X$ and $p_V : \tilde{V} \to Y$ are the canonical maps from the normalization $\tilde{V}$ of V. We extend this linearly to a map $\mathrm{Cor}(X, Y) \to \mathrm{Hom}_R(\mathscr{M}(Y), \mathscr{M}(X))$. It remains to prove that this construction is functorial, i.e. for $\alpha \in \mathrm{Cor}(X, Y)$ and $\beta \in \mathrm{Cor}(Y, Z)$ we have to show $\mathscr{M}(\beta\circ\alpha) = \mathscr{M}(\alpha)\circ\mathscr{M}(\beta)$. We may assume that $\alpha = [V]$ and $\beta = [W]$ are elementary correspondences. Using the same notation as in the proof of Lemma 1.1.5, we get:

$$\begin{aligned}
\mathscr{M}(\alpha)\circ\mathscr{M}(\beta) &= q_{V*}p_V^* q_{W*}p_W^*, && \text{by defn}\\
&= q_{V*}\left(\sum_{C\subseteq \tilde{V}\times_Y\tilde{W}} \mathrm{lg}(\mathscr{O}_{\tilde{V}\times_Y\tilde{W},\eta_C})\cdot q'_{C*}p'^*_C\right) p_W^*, && \text{by (3)}\\
&= \sum_{C\subseteq \tilde{V}\times_Y\tilde{W}} \mathrm{lg}(\mathscr{O}_{\tilde{V}\times_Y\tilde{W},\eta_C})\cdot q_{C*}p_C^* \\
&= \sum_{T\subseteq V\times_Y W}\left(\sum_{C/T}\mathrm{lg}(\mathscr{O}_{\tilde{V}\times_Y\tilde{W},\eta_C})\deg(\eta_C/\eta_T)\right)\cdot q_{T*}p_T^*, && \text{by (2)}\\
&= \sum_{T\subseteq V\times_Y W}\mathrm{lg}(\mathscr{O}_{V\times_Y W,\eta_T})\cdot q_{T*}p_T^*, && \text{by 1.1.5}\\
&= \mathscr{M}(\beta\circ\alpha), && \text{by defn.}
\end{aligned}$$

Hence the statement. ∎

1.3. **Mackey functors with specialization map.**

1.3.1. **Definition.** An *R-Mackey functor* (or just Mackey functor) is an R-linear contravariant functor

$$M : \mathrm{ptCor}^{\mathrm{op}} \to (R - \mathrm{mod}).$$

We denote by $\mathbf{MF}_R = \mathbf{MF}$ the category of Mackey functors with morphisms the natural transformations.

1.3.2. *Notation.* Let $\mathscr{M}$ be a presheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$ (or a Mackey functor). Let $f : X \to Y$ be a flat map in $\mathrm{Reg}^{\leqslant 1}$ (equivalently each connected component of X dominates a connected component of Y), then we define the *pullback* attached to f as

$$f^* := \mathscr{M}([\Gamma_f]) : \mathscr{M}(Y) \to \mathscr{M}(X).$$

Let $f : X \to Y$ be finite and flat (i.e. each connected component of X maps finite and surjective to a connected component of Y), then we define the *pushforward or trace* attached to f as

$$f_* = \mathrm{Tr}_f = \mathrm{Tr}_{X/Y} := \mathscr{M}([\Gamma_f^t]) : \mathscr{M}(X) \to \mathscr{M}(Y).$$

Let $i : P \hookrightarrow U$ be the inclusion of a closed point P in a 1-dimensional scheme in $\mathrm{Reg}^{\leqslant 1}$, then we define the *specialization map* attached to i as

$$s_P := i^* := \mathscr{M}([\Gamma_i]) : \mathscr{M}(U) \to \mathscr{M}(P).$$

Let $C \in (\mathscr{C}/S)$ be a curve, then we denote by $\mathscr{M}_C$ the presheaf on the Zariski site of C induced by $\mathscr{M}$. For $x \in C$ a point we denote by $\mathscr{M}_{C,x}$ the Zariski stalk of $\mathscr{M}_C$ at x and by $\mathscr{M}^h_{C,x}$ the Nisnevich stalk, i.e. the inductive limit of $\mathscr{M}(U)$ over all Nisnevich neighborhoods U of $x \in C$. Notice that $\mathscr{M}_{C,\eta_C} = \mathscr{M}^h_{C,\eta_C}$. Also notice that for $C \in (\mathscr{C}/S)$ and $P \in C$ the specialization map also induces maps (which by abuse of notation we still denote by s_P)

$$s_P : \mathscr{M}_{C,P} \to \mathscr{M}(P), \quad s_P : \mathscr{M}^h_{C,P} \to \mathscr{M}(P).$$

1.3.3. *Remark.* It follows from the proof of Lemma 1.2.2 that a Mackey functor is the same as giving two functors

$$M^* : (\mathrm{pt/S})^{\mathrm{o}} \to (\mathrm{R - mod}), \quad \mathrm{M}_* : (\mathrm{pt/S})_* \to (\mathrm{R - mod}),$$

which satisfy the following two relations:

(MF0) For all $x \in (\mathrm{pt}/S)$ we have $M_*(x) = M^*(x) =: M(x)$.

(MF1) Let $\varphi : y \to x$ be a finite morphism of S-points and $\psi : z \to x$ any morphism of S-points. For $s \in y \times_x z$ denote by $\psi_s : s \to y$ and $\varphi_s : s \to z$ the natural maps induced by the projections and set $l_s := \mathrm{length}(\mathscr{O}_{y\times_x z,s})$. Then

$$M^*(\psi) \circ M_*(\varphi) = \sum_{s \in y \times_x z} l_s \cdot M_*(\varphi_s) \circ M^*(\psi_s) : M(y) \to M(z).$$

(MF2) Let $\varphi : y \to x$ be a finite morphism of S-points, then $M_*(\varphi) \circ M^*(\varphi)$ is multiplication with $\deg(y/x)$.

1.3.4. *Remark.* Notice that there are also different definitions of Mackey functors in the literature (e.g. functoriality is only required for separable field extensions or (MF2) is not required to hold).

1.3.5. **Definition.** An *R-Mackey functor with specialization map* is a Nisnevich sheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$, which additionally satisfies the following two conditions:

(Inj) For all open immersions $j : U \hookrightarrow X$ between connected schemes in $\mathrm{Reg}^{\leqslant 1}$ the restriction map
$$j^* : \mathscr{M}(X) \hookrightarrow \mathscr{M}(U)$$
is injective.

(F.P.) For all $C \in (\mathscr{C}/S)$ with generic point η the natural map
$$\varinjlim_{U \ni \eta} \mathscr{M}(U) \xrightarrow{\simeq} \mathscr{M}(\eta)$$
is an isomorphism, where the limit is over all integral $U \in \mathrm{Reg}^{\leqslant 1}$ of dimension 1 and with generic point η.

We denote by $\mathbf{MFsp}_R = \mathbf{MFsp}$ the full subcategory of $\mathbf{NT}$ with objects the Mackey functors with specialization map.

The name "Mackey functor with specialization map" is justified by the following proposition.

1.3.6. **Proposition.** *A Mackey functor with specialization map is the same as giving a triple $(M, \mathscr{M}, s)$, where*

(1) M is a Mackey functor,

(2) $\mathscr{M}$ is a collection of Zariski sheaves of R-modules $\mathscr{M}_C$ on C, $C \in (\mathscr{C}/S)$, together with a collection of pullback morphisms $\pi^* : \mathscr{M}_C \to \pi_*\mathscr{M}_D$ for each morphism $\pi : D \to C$ in $(\mathscr{C}/S)$ and a collection of pushforward morphisms $\pi_* : \pi_*\mathscr{M}_D \to \mathscr{M}_C$, for each finite morphism $\pi : D \to C$ in $(\mathscr{C}/S)$, which both are functorial.

(3) s is a collection of R-linear homomorphisms $s_P : \mathscr{M}_{C,P} \to M(P)$ for $C \in (\mathscr{C}/S)$ and $P \in C$ (where $\mathscr{M}_{C,P}$ denotes the Zariski stalk).

These data satisfy the following conditions:

(RS0) For all $C \in (\mathscr{C}/S)$ the functor on the small Nisnevich site C_{Nis}, which sends an étale C-scheme U to $\oplus_i \mathscr{M}_{C_i}(U_i)$, where the U_i's are the connected components of U with projective regular model C_i, and which on morphisms is defined using the pullbacks from (2), is a sheaf.

(RS1) For any $C \in (\mathscr{C}/S)$ with generic point η_C, the Zariski-sheaf $\mathscr{M}_C$ is a subsheaf of the constant sheaf $M(\eta_C)$, inducing an isomorphism at the generic stalk $\mathscr{M}_{C,\eta_C} = M(\eta_C)$, which is compatible with the pullback and pushforward morphisms of $\mathscr{M}$ and M.

(RS2) For any 1-dimensional and integral $U \in \mathrm{Reg}^{\leqslant 1}$ with projective model $C \in (\mathscr{C}/S)$ and any map $\varphi : U \to x$ to an S-point x, the pullback $\varphi^* : M(x) \to M(\eta_C)$ has its image contained in $\mathscr{M}_C(U)$.

(S1) For any morphism $\pi : D \to C$ in $(\mathscr{C}/S)$, $Q \in D$, $P := \pi(Q)$ and $a \in \mathscr{M}_{C,P}$
$$\pi_Q^*(s_P(a)) = s_Q(\pi^* a) \quad \text{in } M(Q),$$
where $\pi_Q : Q \to P$ denotes the map induced by π.

(S2) For any finite morphism $\pi : D \to C$ in $(\mathscr{C}/S)$, any $P \in C$ and $a \in \pi_*(\mathscr{M}_D)_P$
$$s_P(\pi_*(a)) = \sum_{Q \in \pi^{-1}(P)} e(Q/P) \cdot \pi_{Q*}(s_Q(a)) \quad \text{in } M(P),$$
where $\pi_Q : Q \to P$ is the finite morphism induced by π and s_Q on the right-hand side is the composition $\pi_*(\mathscr{M}_D)_P \to \mathscr{M}_{D,Q} \xrightarrow{s_Q} M(Q)$.

(S3) For any $C \in (\mathscr{C}/S)$ and any closed point $P \in C$ and $a \in M(x_C)$
$$s_P(\rho_C^*(a)) = \bar{\rho}_C^*(a),$$

where $\bar{\rho}_C : P \to x_C$ is induced by ρ_C. Notice that this makes sense because of (RS2).

Proof. It follows from Lemma 1.1.4, Notation 1.3.2 and Remark 1.3.3 that a Mackey functor with specialization map determines a triple as in the proposition. Conversely given such a triple $(M, \mathscr{M}, s)$ we can construct a Mackey functor with specialization $\tilde{\mathscr{M}}$ map as follows: For an S-point x set $\tilde{\mathscr{M}}(x) := M(x)$ and for $U \in \mathrm{Reg}^{\leqslant 1}$ integral and 1-dimensional set $\tilde{\mathscr{M}}(U) := \mathscr{M}_C(U)$, where $C \in (\mathscr{C}/S)$ is the unique curve which contains $U \subset C$ as an open subset. We extend this additively to all objects in $\mathrm{Reg}^{\leqslant 1}$. If $[Z] \in \mathrm{Cor}(X, Y)$ is an elementary correspondence between integral schemes in $\mathrm{Reg}^{\leqslant 1}$ and $\tilde{Z}$ is the normalization of Z and $q_Z : \tilde{Z} \to X$ and $p_Z : \tilde{Z} \to Y$ are the natural maps induced by the projections, then we define

$$\tilde{\mathscr{M}}([Z]) := \begin{cases} q_{Z*} \circ p_Z^*, & \text{if } p_Z(\tilde{Z}) \text{ contains the generic point of } Y, \\ q_{Z*} \circ \varphi^* \circ s_P, & \text{if } p_Z \text{ factors as } \tilde{Z} \xrightarrow{\varphi} P \text{ with } \{P\} \subsetneq Y \text{ closed}, \end{cases}$$

where the pullback and pushforward on the right hand side are induced by the Mackey functor structure and the structure underlying $\mathscr{M}$ respectively; e.g. if $\tilde{Z}$ is 1-dimensional with projective model C the map $\varphi^* : M(P) \to \mathscr{M}_C(\tilde{Z})$ above is the map induced by $\varphi^* : M(P) \to M(\eta_{\tilde{Z}})$ via (RS2). It is straightforward to check that this defines a Mackey functor with specialization map (cf. the proof of Lemma 1.2.2). □

The main reason we work with the Nisnevich topology is the following Lemma due to V. Voevodsky.

1.3.7. **Lemma.** *Let $\mathscr{M}$ be a presheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$ and denote by $\mathscr{M}_{\mathrm{Nis}}$ the associated sheaf of R-modules in the Nisnevich topology on $\mathrm{Reg}^{\leqslant 1}$. Then $\mathscr{M}_{\mathrm{Nis}}$ uniquely extends to a Nisnevich sheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$ and the canonical map $\mathscr{M} \to \mathscr{M}_{\mathrm{Nis}}$ is a morphism of presheaves with transfer. Moreover the underlying Mackey functors of $\mathscr{M}$ and $\mathscr{M}_{\mathrm{Nis}}$ are equal and if $\mathscr{M}$ satisfies the conditions (Inj.) and (F.P.) from Definition 1.3.5, then $\mathscr{M}_{\mathrm{Nis}}$ is a Mackey functor with specialization map.*

Proof. The first two statements are exactly [35, Lemma 3.1.6] (there it is proven for presheaves with transfers on $Sm\mathrm{Cor}$ but the proof goes through in our situation). That the underlying Mackey functors of $\mathscr{M}$ and $\mathscr{M}_{\mathrm{Nis}}$ are equal follows from the fact that a Nisnevich covering of a point always admits the identity as a refinement. If $\mathscr{M}$ satisfies (F.P.), then the natural map $\varinjlim_{U \ni \eta} \mathscr{M}_{\mathrm{Nis}}(U) \to \mathscr{M}_{\mathrm{Nis}}(\eta) = \mathscr{M}(\eta)$ is surjective, where the limit is over all integral $U \in \mathrm{Reg}^{\leqslant 1}$ of dimension 1 with generic point η. Hence it suffices to show that if $\mathscr{M}$ additionally satisfies (Inj), then $\mathscr{M}_{\mathrm{Nis}}(U) \to \mathscr{M}_{\mathrm{Nis}}(\eta)$, for U as above, is injective. For this take $a \in \mathscr{M}_{\mathrm{Nis}}(U)$, which maps to zero in $\mathscr{M}_{\mathrm{Nis}}(\eta)$ and denote by $\sigma : \mathscr{M} \to \mathscr{M}_{\mathrm{Nis}}$ the canonical map. Then we find a Nisnevich cover $\pi : V \to U$ and an element $b \in \mathscr{M}(V)$ such that $\pi^* a = \sigma(b)$ and $\pi^*(a)$ maps to zero in $\mathscr{M}(\eta) \oplus_i \mathscr{M}(\eta_i)$, where η_i are the generic points $\neq \eta$ of V. Therefore by (Inj.) and (P. F.) $b = 0$, hence also $\pi^* a = 0$ and thus $a = 0$. This yields the statement. □

1.4. **The modulus condition.**

1.4.1. **Definition.** Let $\mathscr{M}$ be a Mackey functor with specialization, $C \in (\mathscr{C}/S)$, $a \in \mathscr{M}(\eta_C)$ and let $\mathfrak{m} = \sum_{P \in C} n_P P$ be an effective divisor on C. Then we say that $\mathfrak{m}$ is a *modulus for a* if and only if the following condition is satisfied:

(MC) $a \in \mathscr{M}(C \setminus |\mathfrak{m}|)$ (this makes sense by (F.P.)) and for $f \in \mathbb{G}_m(\eta_C)$ with $f \equiv 1 \bmod \mathfrak{m}$ we have

$$\sum_{P \in C \setminus |\mathfrak{m}|} v_P(f) \mathrm{Tr}_{P/x_C} s_P(a) = 0,$$

where x_C is defined in (1.1). (Here we allow $\mathfrak{m} = 0$, in which case $f \equiv 1 \bmod \mathfrak{m}$ is an empty condition.) Clearly the set of all $a \in \mathscr{M}(\eta)$, which have $\mathfrak{m}$ as a modulus define an R-submodule, denoted by

$$\mathscr{M}(C, \mathfrak{m}) = \{a \in \mathscr{M}(\eta) \mid \mathfrak{m} \text{ is a modulus for } a\}.$$

We want to give a reformulation of the modulus condition in terms of correspondences.

1.4.2. **Definition.** For two pairs $(C, \mathfrak{m})$ and $(D, \mathfrak{n})$ with $C, D \in (\mathscr{C}/S)$ curves and $\mathfrak{m}$ and $\mathfrak{n}$ effective divisors on them, define

$$\mathrm{Cor}((C, \mathfrak{m}), (D, \mathfrak{n})) \subset \mathrm{Cor}(C \setminus |\mathfrak{m}|, D \setminus |\mathfrak{n}|)$$

to be the free R-module generated by symbols $[V]$, where $V \subset (C \setminus |\mathfrak{m}|) \times (D \setminus |\mathfrak{n}|)$ is an integral closed subscheme which is finite and surjective over $C \setminus |\mathfrak{m}|$, such that $\nu^*(\mathfrak{m} \times D) \leqslant \nu^*(C \times \mathfrak{n})$ with $\nu : \tilde{V} \to \bar{V} \subset C \times D$ the normalization of the closure of V.

1.4.3. **Lemma.** *Let $C \in (\mathscr{C}/S)$ be a curve, $\mathfrak{m}$ an effective divisor on it and $\mathscr{M}$ a Mackey functor with specialization map. Then an element $a \in \mathscr{M}(\eta_C)$ has modulus $\mathfrak{m}$ if and only if $a \in \mathscr{M}(C \setminus |\mathfrak{m}|)$ and for all $\gamma \in \mathrm{Cor}((C, \mathfrak{m}), (\mathbb{P}^1_{x_C}, \{1\}))$ whose components are finite and surjective also over $\mathbb{P}^1_{x_C} \setminus \{1\}$ we have*

$$i_0^* \mathscr{M}(\gamma^t)(a) = i_\infty^* \mathscr{M}(\gamma^t)(a), \tag{1.3}$$

where $i_\epsilon : \{\epsilon\} \hookrightarrow \mathbb{P}^1$, $\epsilon \in \{0, \infty\}$, are the closed immersions.

Proof. We can assume that $\mathfrak{m}$ is not the zero divisor, since else no γ as above exists and the statement reduces to $a \in \mathscr{M}(\eta_C)$ has modulus 0 iff $a \in \mathscr{M}(C)$, which we knew already.

Clearly (1.3) implies (MC): Indeed for $f \in \mathbb{G}_m(\eta_C)$ with $f \equiv 1 \bmod \mathfrak{m}$, take $[\Gamma] \in \mathrm{Cor}(C, \mathbb{P}^1)$ to be the graph of the map $C \to \mathbb{P}^1$ determined by f. Then $[\Gamma]$ restricts to an element γ in $\mathrm{Cor}((C, \mathfrak{m}), (\mathbb{P}^1_{x_C}, \{1\}))$ and in this case (1.3) is just a reformulation of (MC) (use (S2)).

Now assume that a has modulus $\mathfrak{m}$ and let $\gamma = [V] \in \mathrm{Cor}((C, \mathfrak{m}), (\mathbb{P}^1_{x_C}, \{1\}))$ be an elementary correspondence, which is finite and surjective over $\mathbb{P}^1_{x_C} \setminus \{1\}$. Denote by $\tilde{V}$ the normalization of the closure of V in $C \times \mathbb{P}^1_{x_C}$ and let $\pi_C : \tilde{V} \to C$ and $\pi_{\mathbb{P}^1} : \tilde{V} \to \mathbb{P}^1_{x_C}$ be the natural maps induced by projection. Using Lemma 1.1.4, formulas (S1), (S2) and the notations introduced in 1.3.2 we obtain

$$\begin{aligned} i_0^* \mathscr{M}(\gamma^t)(a) - i_\infty^* \mathscr{M}(\gamma^t)(a) = &\sum_{Q \in \pi_{\mathbb{P}^1}^{-1}(0)} e(Q/0) \cdot \pi_{C,Q*} \pi_{\mathbb{P}^1,Q}^* (s_{\pi_C(Q)}(a)) \\ &- \sum_{Q \in \pi_{\mathbb{P}^1}^{-1}(\infty)} e(Q/\infty) \cdot \pi_{C,Q*} \pi_{\mathbb{P}^1,Q}^* (s_{\pi_C(Q)}(a)), \end{aligned}$$

where $\pi_{C,Q} : Q \to \pi_C(Q)$ and $\pi_{\mathbb{P}^1,Q} : Q \to \pi_{\mathbb{P}^1}(Q)$ are the natural maps. Now by (MF2) we have $\pi_{C,Q*} \pi_{\mathbb{P}^1,Q}^* = f(Q/P) \cdot \mathrm{Tr}_{P/x_C}$, with $P = \pi_C(Q)$; further let $f \in \mathbb{G}_m(\eta_{\tilde{V}})$ be the function corresponding to $\pi_{\mathbb{P}^1}$, then $f \equiv 1 \bmod \pi^* \mathfrak{m}$ (by

definition of Cor for pairs) and $e(Q/0) = v_Q(f)$, $e(Q/\infty) = -v_Q(f)$. Thus

$$\begin{aligned} i_0^*\mathscr{M}(\gamma^t)(a) - i_\infty^*\mathscr{M}(\gamma^t)(a) &= \sum_{P\in C\setminus|\mathfrak{m}|}\left(\sum_{Q\in\pi_C^{-1}(P)} f(Q/P)v_Q(f)\right)\cdot s_P(a) \\ &= \sum_{P\in C\setminus|\mathfrak{m}|} v_P(\mathrm{Nm}_{\eta_{\tilde{V}}/\eta_C}(f))\cdot s_P(a). \end{aligned}$$

But this last sum is zero since a has modulus $\mathfrak{m}$ and $\mathrm{Nm}_{\eta_{\tilde{V}}/\eta_C}(f) \equiv 1 \bmod \mathfrak{m}$ by the following Lemma; hence the statement. □

1.4.4. **Lemma.** *Let L/K be a finite field extension and v a discrete valuation on K with valuation ring A. Assume that the normalization B of A in K is a finite A-module and let $w_1, \ldots, w_r$ be all the extensions of v to L. Let $K_s \subset L$ be the separable closure of K in L and $w_{i,s} := w_{i|K_s}$, $i = 1, \ldots, r$. Then for all $m \geqslant 1$ and $n_i \geqslant \frac{m\cdot e(w_{i,s}/v)}{f(w_i/w_{i,s})}$ we have*

$$\mathrm{Nm}_{L/K}(U_{w_1}^{(n_1)} \cap \ldots \cap U_{w_r}^{(n_r)}) \subset U_v^{(m)}.$$

Proof. It suffices to consider separately the two cases in which L/K is either purely inseparable or separable. In the purely inseparable case we have $\mathrm{Nm}_{L/K}(g) = g^{[L:K]}$ and $[L:K] = e(w/v)f(w/v)$. Thus for $g = 1 + \tau^n a$ with $a \in \mathscr{O}_w$, $n \geqslant \frac{m}{f(w/v)}$ and τ a local parameter at w we have $\mathrm{Nm}_{L/K}(g) = 1 + t^m b$ for some $b \in A$ and t a local parameter at v. In the separable case take $g \in U_{w_1}^{(n_1)} \cap \ldots \cap U_{w_r}^{(n_r)}$, $n_i \geqslant me(w_i/v)$, and choose an element $\tau \in B$ which is a local parameter at $w_1, \ldots, w_r$. Then there exist $a_i \in \mathscr{O}_{w_i}$ such that $g = 1 + \tau^{n_i} a_i$. We can rewrite this as $g = 1 + t^m a_i'$ with $a_i' \in \mathscr{O}_{w_i}$. It follows that the a_i' are all equal to an element, say, $a' \in B$. Thus

$$\mathrm{Nm}_{L/K}(g) = \prod_{\sigma\in\mathrm{Hom}_K(L,\bar{K})} (1 + t^n\sigma(a')) \in U_v^{(n)}.$$

Hence the statement. □

Recall that for a curve $C \in (\mathscr{C}/S)$ and $\mathfrak{m}$ an effective divisor on it one denotes by $\mathrm{CH}^1(C, \mathfrak{m})$ the group of divisors on $C \setminus |\mathfrak{m}|$ modulo the group generated by $\mathrm{div} f$, where $f \in \mathbb{G}_m(\eta)$ is a function with $f \equiv 1 \bmod \mathfrak{m}$.

1.4.5. **Proposition.** *Let $\mathscr{M}$ be a Mackey functor with specialization map, $C \in (\mathscr{C}/S)$ and $\mathfrak{m}$ an effective divisor on C.*

(1) There is a biadditive pairing

$$\mathrm{CH}^1(C, \mathfrak{m}) \times \mathscr{M}(C, \mathfrak{m}) \to \mathscr{M}(x_C), \quad (\gamma, a) \mapsto a(\gamma),$$

where for a divisor class $\gamma = \sum_{P\in C\setminus|\mathfrak{m}|} n_P[P]$ we define

$$a(\gamma) := \sum_{P\in C\setminus|\mathfrak{m}|} n_P \cdot \mathrm{Tr}_{P/x_C} s_P(a).$$

(2) If $\mathfrak{m} \leqslant \mathfrak{n}$ on C, then $\mathscr{M}(C, \mathfrak{m}) \subset \mathscr{M}(C, \mathfrak{n})$.

(3) Let $\pi : D \to C$ be a finite morphism in $(\mathscr{C}/S)$ and $\mathfrak{m}$ and $\mathfrak{n}$ effective divisors on C and D, respectively, with $\pi^*\mathfrak{m} \geqslant \mathfrak{n}$. Then the pushforward $\pi_* : \mathscr{M}(\eta_D) \to \mathscr{M}(\eta_C)$ restricts to

$$\pi_* : \mathscr{M}(D, \mathfrak{n}) \to \mathscr{M}(C, \mathfrak{m}).$$

(4) Let $\pi : D \to C$ be a morphism in $(\mathscr{C}/S)$ such that the induced map $x_D \xrightarrow{\simeq} x_C$ is an isomorphism and let $\mathfrak{m}$ and $\mathfrak{n}$ be effective divisors on C and D,

respectively, with $\mathfrak{n} \geqslant \pi^*\mathfrak{m}/[D:C]_{\text{insep}}$*. Then* π *is finite and the pullback* $\pi^* : \mathscr{M}(\eta_C) \to \mathscr{M}(\eta_D)$ *restricts to*

$$\pi^* : \mathscr{M}(C,\mathfrak{m}) \to \mathscr{M}(D,\mathfrak{n}).$$

Proof. (1) follows from the definition of $\mathscr{M}(C,\mathfrak{m})$ and (2) is easy. For (3) take $b \in \mathscr{M}(D,\mathfrak{n})$ and $f \in \mathbb{G}_m(\eta_C)$ with $f \equiv 1 \bmod \mathfrak{m}$. Then π^*f is congruent to 1 modulo $\pi^*\mathfrak{m}$ a fortiori modulo $\mathfrak{n}$ and thus (3) follows from

$$\begin{aligned}\sum_{Q\in\pi^{-1}(P)} &\mathrm{Tr}_{x_D/x_C}(v_Q(\pi^*f)\mathrm{Tr}_{Q/x_D}(s_Q(b)))\\ &= v_P(f)\mathrm{Tr}_{P/x_C}\Big(\sum_{Q\in\pi^{-1}(P)} e(Q/P)\mathrm{Tr}_{Q/P}(s_Q(b))\Big)\\ &= v_P(f)\mathrm{Tr}_{P/x_C}(s_P(\pi_*b)), \qquad (1.4)\end{aligned}$$

where the last equality holds by (S2). Finally, for (4) take $a \in \mathscr{M}(C,\mathfrak{m})$ and $g \in \mathbb{G}_m(\eta_D)$ with $g \equiv 1 \bmod \mathfrak{n}$. Then $\pi_*(g) \equiv 1 \bmod \mathfrak{m}$ (by Lemma 1.4.4) and (4) follows from

$$\begin{aligned}\sum_{Q\in\pi^{-1}(P)} v_Q(g)\mathrm{Tr}_{Q/x_D}s_Q(\pi^*a) &= \sum_{Q\in\pi^{-1}(P)} f(Q/P)v_Q(g)\mathrm{Tr}_{P/x_C}(s_P(a))\\ &= v_P(\pi_*g)\mathrm{Tr}_{P/x_C}(s_P(a)), \qquad (1.5)\end{aligned}$$

where the first equality holds by (S1) and (MF2) and the second equality follows from $\mathrm{div}(\pi_*g) = \pi_*\mathrm{div}(g)$. □

Associated symbols.

1.4.6. **Lemma.** *Let* C *be a regular affine connected curve over a field,* $P_1,\ldots,P_r \in C$ *distinct closed points,* $n_1,\ldots,n_r \in \mathbb{N}$ *natural numbers and* $f_1,\ldots,f_r \in \kappa(C)^\times$ *non-zero functions. Then there exists a function* $f \in \kappa(C)^\times$ *such that* $f/f_i \in U_{P_i}^{(n_i)}$ *for all* $i \in \{1,\ldots,r\}$ *and* $f \in \mathscr{O}_{C,P}$ *for all* $P \notin \{P_1,\ldots,P_r\}$.

Proof. By the Approximation Lemma we find $f \in \kappa(C)^\times$ with $v_{P_i}(f_i - f) \geqslant v_{P_i}(f_i) + n_i$ and $v_P(f) \geqslant 0$ if $P \neq P_i$. Since $n_i \geqslant 1$ we in particular get $v_{P_i}(f) = v_P(f_i)$ and thus $v_{P_i}(f_i/f - 1) \geqslant n_i$. □

1.4.7. **Proposition** (cf. [31, III, §1, Prop. 1])**.** *Let* $M_* : (\mathrm{pt}/S)_* \to (R-mod)$ *be a functor and let* $C \in (\mathscr{C}/S)$ *be fixed. Assume for all* $P \in C$ *we are given submodules* $M_P(\eta) \subset M(\eta)$ *and* R*-linear maps* $s_P : M_P(\eta) \to M(P)$*. For* $A \subset C$ *we set* $M_A(\eta) := \cap_{P\in A}M_P(\eta)$ *and for* $\varphi : x \to y$ *in* $(\mathrm{pt}/S)_*$ *we set* $\mathrm{Tr}_{x/y} := M_*(\varphi)$*. Then for any* $a \in M(\eta)$ *the following two statements are equivalent:*

(1) *There exists a family of continuous group homomorphisms* $\{\rho_P : \mathbb{G}_m(\eta) \to M(x_C)\}_{P\in C}$ *and a non-empty open subset* $V \subset C$ *such that*
   (a) $a \in M_V(\eta)$.
   (b) $\rho_P(f) = v_P(f)\mathrm{Tr}_{P/x_C}(s_P(a))$ *for all* $f \in \mathbb{G}_m(\eta)$ *and* $P \in V$.
   (c) $\sum_{P\in C}\rho_P(f) = 0$ *for all* $f \in \mathbb{G}_m(\eta)$.
(2) *There exists an effective divisor* $\mathfrak{m} = \sum_{Q\in C} n_Q Q$ *on* C *such that* $a \in M_{C\setminus|\mathfrak{m}|}(\eta)$ *and for all* $f \in \mathbb{G}_m(\eta)$ *with* $f \equiv 1 \bmod \mathfrak{m}$*, we have*

$$\sum_{P\in C\setminus|\mathfrak{m}|} v_P(f)\mathrm{Tr}_{P/x_C}(s_P(a)) = 0.$$

Furthermore the family $\{\rho_P\}_{P\in C}$ *is uniquely determined.*

Proof. The proof is along the lines of the proof of [31, III, §1, Prop. 1]. Let us first see that if a family $\{\rho_P\}_{P\in C}$ as in (1) exists, then it is unique. Indeed for $P \in V$ nothing is to show. Thus assume $P \notin C$. The continuity of the ρ's implies,

that for all $Q \in C$ we have $\rho_Q(U_Q^{(n_Q)}) = 0$, for some large enough natural numbers n_Q. Given $f \in \mathbb{G}_m(\eta)$ use Lemma 1.4.6 to find a function $f_P \in \mathbb{G}_m(\eta)$ such that $f/f_P \in U_P^{(n_P)}$ and $f_P \in U_Q^{(n_Q)}$ for all $Q \in C \setminus (V \cup \{P\})$. Then by (b) and (c)

$$\rho_P(f) = \sum_{Q \in C \setminus V} \rho_Q(f_P) = -\sum_{Q \in V} v_Q(f_P)\mathrm{Tr}_{Q/x_C}(s_Q(a)),$$

which yields uniqueness.

For (1) $\Rightarrow$ (2) choose $n_Q \geqslant 1$ as above and set $\mathfrak{m} = \sum_{Q \in C \setminus V} n_Q Q$. Then (b) and (c) of (1) immediately imply the equality in (2).

Next (2) $\Rightarrow$ (1). Write $\mathfrak{m} = \sum_i n_i P_i$ with $n_i \geqslant 1$ and set $V := C \setminus |\mathfrak{m}|$. For $P \in C$ and $f \in \mathbb{G}_m(\eta)$ we define ρ_P as follows

$$\rho_P(f) := \begin{cases} v_P(f)\mathrm{Tr}_{P/x_C}(s_P(a)), & \text{if } P \in V \\ -\sum_{Q \in V} v_Q(f_i)\mathrm{Tr}_{Q/x_C}(s_Q(a)), & \text{if } P = P_i, i \in \{1, \dots r\}, \end{cases}$$

where f_i is chosen so that $f_i \in U_{P_j}^{(n_j)}$ for $j \neq i$ and $f/f_i \in U_{P_i}^{(n_i)}$. This definition does not depend on the choices of the f_i's. Indeed if we choose different functions g_i with the same property, then $g_i/f_i \equiv 1 \bmod \mathfrak{m}$ for all i and (2) yields

$$-\sum_{Q \in V} v_Q(f_i)\mathrm{Tr}_{Q/x_C}(s_Q(a)) = -\sum_{Q \in V} v_Q(g_i)\mathrm{Tr}_{Q/x_C}(s_Q(a)).$$

Clearly the function $\rho_P : \mathbb{G}_m(\eta) \to M(x_C)$ thus defined is a group homomorphism, which satisfies $\rho_P(\mathscr{O}_{C,P}^{\times}) = 0$, if $P \in V$, and $\rho_P(U_{P_i}^{(n_i)}) = 0$, if $P = P_i$, therefore it is also continuous. It remains to check condition (c). For this take $f \in \mathbb{G}_m(\eta)$ and choose f_i's as above. Set $h := \prod_i f_i$. Then $f/h \equiv 1 \bmod \mathfrak{m}$ and we obtain

$$\begin{aligned} \sum_i \rho_{P_i}(f) &= -\sum_i \sum_{Q \in V} v_Q(f_i)\mathrm{Tr}_{Q/x_C}(s_Q(a)) && \text{by definition} \\ &= -\sum_{Q \in V} v_Q(h)\mathrm{Tr}_{P/x_C}(s_Q(a)) \\ &= -\sum_{Q \in V} v_Q(f)\mathrm{Tr}_{P/x_C}(s_Q(a)) && \text{by (2)} \\ &= -\sum_{Q \in V} \rho_Q(f) && \text{by definition.} \end{aligned}$$

Hence (c). □

The modulus condition is local in the Nisnevich topology.

1.4.8. **Theorem.** *Let $\mathscr{M} \in$ **MFsp** be a Mackey functor with specialization map. Let $C \in (\mathscr{C}/S)$ be a curve, $U \subset C$ a non-empty open subset and $a \in \mathscr{M}(U)$ a section. Assume there exists an étale cover $\pi : \coprod_i V_i \to U$ satisfying the following properties:*

(1) The V_i's are connected. We denote by $\pi_i : D_i \to C$ in $(\mathscr{C}/S)$ the compactification of $\pi_{|V_i} : V_i \to U$.
(2) There exists an i_0 such that $\pi_{|V_{i_0}} : V_{i_0} \hookrightarrow U$ is an open immersion.
(3) For all i there exists an effective divisors $\mathfrak{n}_i$ on D_i such that $|\mathfrak{n}_i| = D_i \setminus V_i$ and $\pi_i^* a \in \mathscr{M}(D, \mathfrak{n}_i)$.

Then there exists an effective divisor $\mathfrak{m}$ on C such that $|\mathfrak{m}| = C \setminus U$ and $a \in \mathscr{M}(C, \mathfrak{m})$.

In particular the property for a having a modulus $\mathfrak{m}$ with $|\mathfrak{m}| = C \setminus U$ is local in the Nisnevich topology on U.

Before we start with the proof of the theorem we need some preparations.

1.4.9. **Lemma.** *Let $\mathscr{M} \in \mathbf{MFsp}$ be a Mackey functor with specialization map. Let $C \in (\mathscr{C}/S)$ be a curve, $U \subset C$ a non-empty open subset and $a \in \mathscr{M}(U)$ a section. Assume there exist a Zariski open covering $U = \cup_i U_i$ and effective divisors $\mathfrak{m}_i$ on C with $|\mathfrak{m}_i| = C \setminus U_i$ and $a \in \mathscr{M}(C, \mathfrak{m}_i)$, for all i. Then there exists an effective divisor $\mathfrak{m}$ on C with $|\mathfrak{m}| = C \setminus U$ and $a \in \mathscr{M}(C, \mathfrak{m})$.*

Proof. It suffices to consider finite coverings and by induction over the number of $U_i's$ it suffices to consider coverings by two open subsets $U = U_1 \cup U_2$. By Proposition 1.4.5, (2) we can further assume that

$$\mathfrak{m}_1 = \sum_{P \in C \setminus U_1} n[P], \quad \mathfrak{m}_2 = \sum_{P \in C \setminus U_2} n[P],$$

for some large enough positive integer n. Set

$$\mathfrak{m} := \min\{\mathfrak{m}_1, \mathfrak{m}_2\}.$$

Then $|\mathfrak{m}| = |\mathfrak{m}_1| \cap |\mathfrak{m}_2| = (C \setminus U_1) \cap (C \setminus U_2) = C \setminus U$ and we claim

$$a \in \mathscr{M}(C, \mathfrak{m}). \tag{1.6}$$

For this set $\mathfrak{m}_{12} := \max\{\mathfrak{m}_1, \mathfrak{m}_2\}$. Then $C \setminus |\mathfrak{m}_{12}| = U_1 \cap U_2$ and $\mathfrak{m}_{1,2} \geqslant \mathfrak{m}_i$, $i = 1, 2$. Hence

$$a \in \mathscr{M}(C, \mathfrak{m}_1),\ \mathscr{M}(C, \mathfrak{m}_2),\ \mathscr{M}(C, \mathfrak{m}_{12}). \tag{1.7}$$

Therefore the pairs $(a, \mathfrak{m}_1)$, $(a, \mathfrak{m}_2)$, $(a, \mathfrak{m}_{12})$ satisfy condition (2) of Proposition 1.4.7 (with $M_* := \mathscr{M}_{|(\mathrm{pt}/S)}$ and $M_P(\eta_C) = \mathscr{M}_{C,P}$). Thus we get families of continuous group homomorphisms $\{\rho_{P,\mathfrak{m}_*} : \mathbb{G}_m(\eta_C) \to \mathscr{M}(x_C)\}$ satisfying the properties (a)-(c) from Proposition 1.4.7, (1) (with $V = C \setminus |\mathfrak{m}_*|$), for $* \in \{1, 2, 12\}$. But the uniqueness statement of this proposition yields

$$\rho_{P,\mathfrak{m}_1} = \rho_{P,\mathfrak{m}_{12}} = \rho_{P,\mathfrak{m}_2} =: \rho_P. \tag{1.8}$$

Now fix $f \in \mathbb{G}_m(\eta_C)$ with $f \equiv 1 \bmod \mathfrak{m}$. Take $N \geqslant n$ with $\rho_P(U_P^{(N)}) = 0$ for all $P \in \mathfrak{m}_{12}$ (exists by continuity) and choose $h \in \mathbb{G}_m(\eta_C)$ so that

$$\frac{h}{f} \in U_P^{(N)} \text{ for all } P \in |\mathfrak{m}| \quad \text{and} \quad h \in U_P^{(N)} \text{ for all } P \in |\mathfrak{m}_{12}| \setminus |\mathfrak{m}|.$$

Then

(1) $h \equiv 1 \bmod \mathfrak{m}_*$, for all $* \in \{1, 2, 12\}$.
(2) $\sum_{P \in |\mathfrak{m}_*|} \rho_P(h) = 0$, for all $* \in \{1, 2, 12\}$.
(3) $\rho_P(h) = \rho_P(f)$ for all $P \in |\mathfrak{m}|$.
(4) $\rho_P(f) = v_P(f)\mathrm{Tr}_{P/x_C}(s_P(a))$, for all $P \in U$.

Here (1) holds by definition, (2) follows from (1.7), (1.8) and reciprocity, (3) from the choice of N and h and (4) from (1.8) and $U = U_1 \cup U_2$. Hence

$$\begin{aligned}
0 &= \sum_{P \in \mathfrak{m}_{12}} \rho_P(h) - \sum_{P \in \mathfrak{m}_1} \rho_P(h) - \sum_{P \in \mathfrak{m}_2} \rho_P(h), && \text{by (2)} \\
&= -\sum_{P \in \mathfrak{m}} \rho_P(h) = -\sum_{P \in \mathfrak{m}} \rho_P(f), && \text{by (3)} \\
&= \sum_{P \in U} \rho_P(f), && \text{by reciprocity} \\
&= \sum_{P \in U} v_P(f)\mathrm{Tr}_{P/x_C}(s_P(a)), && \text{by (4).}
\end{aligned}$$

This proves the Claim (1.6) and hence the lemma. □

1.4.10. *Notation.* For $C \in (\mathscr{C}/S)$ with function field $K = \kappa(C)$ and $\mathfrak{m}$ an effective divisor on C we denote

$$K_{\mathfrak{m},1} := \{f \in K^\times \mid f \equiv 1 \bmod \mathfrak{m}\}.$$

Notice that $K_{\mathfrak{m},1} \subset K^\times$ is a subgroup and that $K_{\mathfrak{m}',1} \subset K_{\mathfrak{m},1}$ for all effective divisors $\mathfrak{m} \leqslant \mathfrak{m}'$ and $K_{\{0\},1} = K^\times$.

1.4.11. **Lemma.** *Let K be the function field of a curve $C \in (\mathscr{C}/S)$, $\mathfrak{m}$ an effective divisor on C and $P \in C \setminus |\mathfrak{m}|$ a closed point. Then for all $n \geqslant 1$ the composition of the natural maps $K_{\mathfrak{m},1} \hookrightarrow K^\times \twoheadrightarrow K^\times/U_P^{(n)}$ induces an isomorphism of groups*

$$\frac{K_{\mathfrak{m},1}}{K_{\mathfrak{m}+n[P],1}} \xrightarrow{\simeq} \frac{K^\times}{U_P^{(n)}}.$$

Proof. Clearly the natural map $K_{\mathfrak{m}} \to K^\times/U_P^{(n)}$ factors to give the map $\tau : K_{\mathfrak{m},1}/K_{\mathfrak{m}+n[P],1} \to K^\times/U_P^{(n)}$ from the statement. Now for $a \in K^\times/U_P^{(n)}$ take a lift $\tilde{a} \in K^\times$ and choose $\tilde{a}_P \in K^\times$ with $\tilde{a}_P/\tilde{a} \in U_P^{(n)}$ and $\tilde{a}_P \in U_Q^{(m_Q)}$ for all $Q \in |\mathfrak{m}|$, where $\mathfrak{m} = \sum_{Q \in C} m_Q[Q]$. Then $\tilde{a}_P \in K_{\mathfrak{m},1}$ and we set $\sigma(a) :=$ class of $\tilde{a}_P$ in $K_{\mathfrak{m},1}/K_{\mathfrak{m}+n[P],1}$. It is straightforward to check that $\sigma(a)$ is independent of all the choices and that we obtain a group homomorphism $\sigma : K^\times/U_P^{(n)} \to K_{\mathfrak{m},1}/K_{\mathfrak{m}+n[P],1}$, which is inverse to τ. Hence the lemma. □

1.4.12. **Lemma.** *Let $\pi : D \to C$ be a finite flat morphism in $(\mathscr{C}/S)$. Let $\mathfrak{m}$ be an effective divisor on C and $P \in C \setminus |\mathfrak{m}|$ a closed point and denote by $L = \kappa(D)$ and $K = \kappa(C)$ the function fields. Assume there exists a closed point $Q \in \pi^{-1}(P)$ such that π is étale in a neighborhood of Q (in particular L/K is separable). Then for all $n \geqslant 1$ the norm map $\mathrm{Nm} : L^\times \to K^\times$ induces a surjective group homomorphism*

$$\mathrm{Nm} : \frac{L_{\pi^*\mathfrak{m},1}}{L_{\pi^*\mathfrak{m}+n[Q],1}} \twoheadrightarrow \frac{K_{\mathfrak{m},1}}{K_{\mathfrak{m}+n[P],1}}.$$

Proof. Denote by $\hat{L}$ and $\hat{K}$ the completions of L and K with respect to v_Q and v_P, respectively. Denote by $\hat{U}_Q^{(n)}$ and $\hat{U}_P^{(n)}$ the corresponding groups of higher 1-units. Then $\hat{L}$ is separable over $\hat{K}$ as well as $\kappa(Q)$ over $\kappa(P)$ and v_Q is unramified over v_P. Thus the norm map $\widehat{\mathrm{Nm}} : \hat{L} \to \hat{K}$ is surjective and induces by Lemma 1.4.4 a surjective group homomorphism

$$\widehat{\mathrm{Nm}} : \frac{\hat{L}^\times}{\hat{U}_Q^{(n)}} \twoheadrightarrow \frac{\hat{K}^\times}{\hat{U}_P^{(n)}}.$$

Furthermore, $\mathrm{Nm} : L^\times \to K^\times$ induces by Lemma 1.4.4 a map

$$\mathrm{Nm} : L_{\pi^*\mathfrak{m},1} \to K_{\mathfrak{m},1}.$$

Since the diagram with horizontal maps the natural inclusions

$$\begin{tikzcd} L^\times \arrow[r] \arrow[d, "\mathrm{Nm}"'] & \hat{L}^\times \arrow[d, "\widehat{\mathrm{Nm}}"] \\ K^\times \arrow[r] & \hat{K}^\times \end{tikzcd}$$

commutes, Lemma 1.4.11 gives the following commutative diagram

$$\begin{tikzcd} L_{\pi^*\mathfrak{m},1} \arrow[r, two heads] \arrow[d, "\mathrm{Nm}"'] & \frac{L_{\pi^*\mathfrak{m},1}}{L_{\pi^*\mathfrak{m}+n[Q],1}} \arrow[r, "\simeq"] & \frac{\hat{L}^\times}{\hat{U}_Q^{(n)}} \arrow[d, "\widehat{\mathrm{Nm}}"] \\ K_{\mathfrak{m},1} \arrow[r, two heads] & \frac{K_{\mathfrak{m},1}}{K_{\mathfrak{m}+n[P],1}} \arrow[r, "\simeq"] & \frac{\hat{K}^\times}{\hat{U}_P^{(n)}}. \end{tikzcd}$$

This yields the statement. □

Proof of Theorem 1.4.8. By Lemma 1.4.9 we are reduced to prove the following: Assume there exists a finite flat map $\pi : D \to C$ in $(\mathscr{C}/S)$ and effective divisors $\mathfrak{m}'$ and $\mathfrak{n}$ on C and D, respectively such that

- $U' := C \setminus |\mathfrak{m}'|$ is strictly contained in U.
- π restricted to $V := D \setminus |\mathfrak{n}|$ is étale over U.
- $U = U' \cup \pi(V)$.
- $a \in \mathscr{M}(C, \mathfrak{m}')$ and $\pi^* a \in \mathscr{M}(D, \mathfrak{n})$.

Then we have to show, that there exists an effective divisor $\mathfrak{m}$ on C such that $U = C \setminus |\mathfrak{m}|$ and $a \in \mathscr{M}(C, \mathfrak{m})$. We distinguish two cases:

First case: $U \setminus U' = \{P_0\}$. By Lemma 1.4.9 and Proposition 1.4.5, (2) we can make U smaller around P_0 and $\mathfrak{m}'$ and $\mathfrak{n}$ bigger. Thus we can assume

(1) $V = \pi^{-1}(U') \cup \{Q_0\}$, where $Q_0 \in \pi^{-1}(P_0)$ and π is étale in a neighborhood of Q_0.
(2) $\mathfrak{m}' = \mathfrak{m} + n[P_0]$, with $\mathfrak{m}$ an effective divisor on C with $P_0 \notin |\mathfrak{m}|$, i.e. $|\mathfrak{m}| = C \setminus U$.
(3) $\pi^*\mathfrak{m}' \geqslant \mathfrak{n}$.

Since $|\mathfrak{n}| = D \setminus V = \pi^{-1}(C \setminus U') \setminus \{Q_0\} = \pi^{-1}(|\mathfrak{m}'|) \setminus \{Q_0\}$ and since the multiplicity of Q_0 in $\pi^*\mathfrak{m}'$ is n (Q_0 being unramified over P_0) we have $\pi^*\mathfrak{m}' - n[Q_0] \geqslant \mathfrak{n}$ and thus we can in fact assume

(4) $\mathfrak{n} = \pi^*\mathfrak{m}' - n[Q_0]$.

We claim

$$a \in \mathscr{M}(C, \mathfrak{m}). \tag{1.9}$$

For this take $f \in \mathbb{G}_m(\eta_C)$ with $f \equiv 1 \mod \mathfrak{m}$. Now by Lemma 1.4.11 we have canonical isomorphisms (notice that $Q_0 \notin |\mathfrak{n}|, |\pi^*\mathfrak{m}|$)

$$\frac{L_{\pi^*\mathfrak{m}'-n[Q_0],1}}{L_{\pi^*\mathfrak{m}',1}} = \frac{L_{\mathfrak{n},1}}{L_{\mathfrak{n}+n[Q_0],1}} \simeq \frac{L^\times}{U_{Q_0}^{(n)}} \simeq \frac{L_{\pi^*\mathfrak{m},1}}{L_{\pi^*\mathfrak{m}+n[Q_0],1}}.$$

Hence by Lemma 1.4.12 the norm map induces a surjection

$$L_{\mathfrak{n},1} \twoheadrightarrow \frac{K_{\mathfrak{m},1}}{K_{\mathfrak{m}+n[P_0],1}} = \frac{K_{\mathfrak{m},1}}{K_{\mathfrak{m}',1}}.$$

Thus there exists a $g \in \mathbb{G}_m(\eta_D)$ with $g \equiv 1 \mod \mathfrak{n}$ and $h \in \mathbb{G}_m(\eta_C)$ with $h \equiv 1 \mod \mathfrak{m}'$ such that

$$\pi_* g = fh \quad \text{in } \mathbb{G}_m(\eta_C).$$

We get

$$\sum_{P\in U} v_P(f)\mathrm{Tr}_{P/x_C}(s_P(a)) = \sum_{P\in U} v_P(f)\mathrm{Tr}_{P/x_C}(s_P(a)) + \sum_{P\in U'} v_P(h)\mathrm{Tr}_{P/x_C}(s_P(a)) \tag{1.10}$$

$$= \sum_{P\in U} v_P(fh)\mathrm{Tr}_{P/x_C}(s_P(a)) \tag{1.11}$$

$$= \sum_{P\in U} v_P(\pi_* g)\mathrm{Tr}_{P/x_C}(s_P(a))$$

$$= \sum_{P\in U}\sum_{Q\in\pi^{-1}(P)} [Q:P]v_Q(g)\mathrm{Tr}_{P/x_C}(s_P(a)) \tag{1.12}$$

$$= \sum_{Q\in\pi^{-1}(U)} v_Q(g)\mathrm{Tr}_{Q/x_C}(s_Q(\pi^* a)) \tag{1.13}$$

$$= \sum_{Q\in V} v_Q(g)\mathrm{Tr}_{Q/x_C}(s_Q(\pi^* a)) \tag{1.14}$$

$$= 0, \tag{1.15}$$

where (1.10) holds since $a \in \mathscr{M}(C,\mathfrak{m}')$, (1.11) by $v_{P_0}(h) = 0$, (1.12) by $\mathrm{div}(\pi_* g) = \pi_*\mathrm{div}(g)$, (1.13) by (MF2), (S1), (1.14) since $v_Q(g) = 0$ for $Q \in \pi^{-1}(P_0)\setminus\{Q_0\}$ and (1.15) holds since $\pi^* a \in \mathscr{M}(D,\mathfrak{n})$. This proves the claim (1.9) and hence the first case.

Second case: $U\setminus U' = \{P_1,\dots,P_r\}$ *for some* $r \geqslant 1$. This is the general case. Set $U_1 := U'\cup\{P_1\}$, $V_1 := \pi^{-1}(U_1)$, $\mathfrak{n}_1 := \mathfrak{n} + \sum_{Q\in V\setminus V_1}[Q]$. Then $|\mathfrak{n}_1| = C\setminus V_1$ and $\pi^* a \in \mathscr{M}(D,\mathfrak{n}_1)$ (since $\mathfrak{n}_1 \geqslant \mathfrak{n}$), π restricted to V_1 is étale, $U_1\setminus U' = \{P_1\}$ and $U_1 = U'\cup\pi(V_1)$. Thus we can apply the first case to get an effective divisor $\mathfrak{m}_1$ on C with $|\mathfrak{m}_1| = C\setminus U_1$ and $a\in\mathscr{M}(C,\mathfrak{m}_1)$. Therefore we can replace $(U',\mathfrak{m}')$ by $(U_1,\mathfrak{m}_1)$ and since $U\setminus U_1 = \{P_2,\dots,P_r\}$ we conclude by induction. □

1.5. Reciprocity functors.

1.5.1. **Definition.** An *R-reciprocity functor* (or just reciprocity functor) is an R-Mackey functor with specialization map $\mathscr{M}$ such that for any $C\in(\mathscr{C}/S)$, any non-empty open subset $U\subset C$ and any section $a\in\mathscr{M}(U)$ there exists an effective divisor $\mathfrak{m}$ on C which has support equal to $C\setminus U$ and is a modulus for a, i.e. for all $\emptyset\neq U\subset C$ we have

$$\mathscr{M}(U) = \bigcup_{|\mathfrak{m}|=C\setminus U}\mathscr{M}(C,\mathfrak{m}), \tag{1.16}$$

where the union is over all effective divisors on C with support equal to $C\setminus U$. (In particular we have $\mathscr{M}(C) = \mathscr{M}(C,0)$ and $\mathscr{M}(\eta_C) = \bigcup_{\mathfrak{m}}\mathscr{M}(C,\mathfrak{m})$, where the union is over all effective divisors on C.)

We denote by $\mathbf{RF}_R = \mathbf{RF}$ the full subcategory of **MFsp** whose objects are reciprocity functors.

1.5.2. *Remark.* Let $\mathscr{M}$ be a Mackey functor with specialization map. By Theorem 1.4.8 the condition (1.16) is satisfied for all $C\in(\mathscr{C}/S)$ and all non-empty open subsets $U\subset C$ if and only if for all $C\in(\mathscr{C}/S)$ and all points $x\in C$ (closed or not) we have

$$\mathscr{M}^h_{C,x} = \varinjlim_{(D,\mathfrak{n})}\mathscr{M}(D,\mathfrak{n}), \tag{1.17}$$

where the limit is over all pairs $(D,\mathfrak{n})$ with $D\to C$ finite flat in $(\mathscr{C}/S)$ and $\mathfrak{n}$ is an effective divisor on D such that $D\setminus|\mathfrak{n}|$ is a Nisnevich neighborhood of x (i.e. it is étale over C and there is a point $y\in D\setminus|\mathfrak{n}|$, which maps isomorphically to x).

1.5.3. **Corollary-Definition.** *Let $\mathscr{M}$ be a reciprocity functor. Then for any $C \in (\mathscr{C}/S)$ and $P \in C$ there exists a biadditive pairing*

$$(-,-)_P : \mathscr{M}(\eta_C) \times \mathbb{G}_m(\eta_C) \to \mathscr{M}(x_C),$$

which is R-linear in the first argument and has the following properties:

(1) *$(-,-)_P$ is continuous when $\mathscr{M}(\eta_C)$ and $\mathscr{M}(x_C)$ are endowed with the discrete topology and $\mathbb{G}_m(\eta_C)$ with the topology for which $\{U_P^{(n)} \mid n \geqslant 1\}$ is a fundamental system of open neighborhoods of 1.*

(2) *For all $a \in \mathscr{M}_{C,P}$ (notation as in 1.3.2) and $f \in \mathbb{G}_m(\eta_C)$ we have*

$$(a, f)_P = v_P(f)\mathrm{Tr}_{P/x_C}(s_P(a)).$$

(3) *For all $a \in \mathscr{M}(\eta_C)$ and $f \in \mathbb{G}_m(\eta_C)$ we have*

$$\sum_{P \in C} (a, f)_P = 0.$$

Furthermore $(-,-)_P$ is uniquely determined by the properties above. We call $(-,-)_P$ the local symbol at P attached to $\mathscr{M}$.

Proof. The Zariski-stalk $\mathscr{M}_{C,P}$ is a submodule of $\mathscr{M}(\eta_C)$ and since $\mathscr{M}_C$ is a Zariski sheaf on C, we have $\mathscr{M}_C(U) = \cap_{P\in U}\mathscr{M}_{C,P}$ for all open subsets $U \subset C$. Thus we are in the situation of Proposition 1.4.7. For $a \in \mathscr{M}(\eta_C)$ choose a modulus $\mathfrak{m}$, then condition (2) in Proposition 1.4.7 is satisfied. Let $\{\rho_{a,P} : \mathbb{G}_m(\eta_C) \to \mathscr{M}(x_C)\}_{P\in C}$ be the family of continuous homomorphisms from Proposition 1.4.7, (1) constructed for $(a, \mathfrak{m})$. Notice that this family does not depend on the choice of the modulus $\mathfrak{m}$ for a: Indeed if $\mathfrak{m}'$ is another modulus for a then so is $\mathfrak{m}'' = \mathfrak{m} + \mathfrak{m}'$. But the family $\{\rho''_{a,P}\}$ constructed via $\mathfrak{m}''$ satisfies in particular, the same properties as $\{\rho_{P,a}\}$ and so by uniqueness they have to be the same. Then for $a \in \mathscr{M}(\eta_C)$ and $f \in \mathbb{G}_m(\eta_C)$ we set

$$(a, f)_P := \rho_{a,P}(f).$$

It follows from the uniqueness of the ρ's that this defines a biadditive pairing, which is R-linear in the first argument. The properties (1)-(3) now follow immediately from the corresponding property of the ρ's. □

1.5.4. **Corollary** (cf. [31, III,§1, Prop. 2])**.** *Let $\Phi : \mathscr{M} \to \mathscr{N}$ be a morphism between reciprocity functors. Then for all $C \in (\mathscr{C}/S)$, $P \in C$, $a \in \mathscr{M}(\eta_C)$ and $f \in \mathbb{G}_m(\eta_C)$ we have*

$$\Phi((a, f)_P^{\mathscr{M}}) = (\Phi(a), f)_P^{\mathscr{N}}.$$

Proof. Follows from the uniqueness statement in Proposition 1.4.7. □

1.5.5. **Proposition** (cf. [31, III, Prop. 3 and 4])**.** *Let $\mathscr{M}$ be a reciprocity functor and $\pi : D \to C$ a finite morphism in $(\mathscr{C}/S)$.*

(1) *For all $b \in \mathscr{M}(\eta_D)$, $f \in \mathbb{G}_m(\eta_C)$ and $P \in C$ we have in $\mathscr{M}(x_C)$*

$$(\pi_* b, f)_P = \sum_{Q \in \pi^{-1}(P)} \mathrm{Tr}_{x_D/x_C}(b, \pi^* f)_Q.$$

(2) *For all $a \in \mathscr{M}(\eta_C)$, $g \in \mathbb{G}_m(\eta_D)$ and $P \in C$ we have in $\mathscr{M}(x_C)$*

$$(a, \pi_* g)_P = \sum_{Q \in \pi^{-1}(P)} \mathrm{Tr}_{x_D/x_C}(\pi^* a, g)_Q.$$

Proof. (1). For $f \in \mathbb{G}_m(\eta_C)$ and $P \in C$ set

$$\rho_P(f) := \sum_{Q \in \pi^{-1}(P)} \mathrm{Tr}_{x_D/x_C}((b, \pi^* f)_Q).$$

Then $\rho_P : \mathbb{G}_m(\eta_C) \to \mathscr{M}(x_C)$ is clearly a group homomorphism, it is continuous since the $(-,-)_Q$'s are and by 1.5.3, (3) we have $\sum_{P\in C}\rho_P(f) = 0$ for all $f \in \mathbb{G}_m(\eta_C)$. Further, let $\mathfrak{n}$ be a modulus for b. Then $\mathfrak{m} := \pi_*\mathfrak{n}$ is a modulus for $\pi_*(b)$ by Proposition 1.4.5, (3) and for $P \in C \setminus |\mathfrak{m}|$ we have $\pi^{-1}(P) \subset D \setminus |\mathfrak{n}|$. Thus as in (1.4) we have $\rho_P(f) = v_P(f)\mathrm{Tr}_{P/x_C}(s_P(\pi_*b))$, for $P \in C \setminus |\mathfrak{m}|$. Hence the family $\{\rho_P\}_{P\in C}$ satisfies the conditions (a)-(c) from Proposition 1.4.7, (1) and thus the statement of (1) follows from the uniqueness statement in Proposition 1.4.7.

(2). Let $\mathfrak{m}$ be a modulus for a. Then $\mathfrak{n} := \pi^*\mathfrak{m}$ is a modulus for π^*a, by Proposition 1.4.5, (4). Thus for $P \in C \setminus |\mathfrak{m}|$ we have as in (1.5)

$$\begin{aligned}&(a, \pi_*g)_P = v_P(\pi_*g)\mathrm{Tr}_{P/x_C}(s_P(a))\\ &= \sum_{Q\in\pi^{-1}(P)} \mathrm{Tr}_{x_D/x_C}(v_Q(g)\mathrm{Tr}_{Q/x_D}(\pi^*a)) = \sum_{Q\in\pi^{-1}(P)} \mathrm{Tr}_{x_D/x_C}((\pi^*a, g)_Q). \qquad (1.18)\end{aligned}$$

Now we consider the case $P \in |\mathfrak{m}|$. For $P' \in |\mathfrak{m}|$ choose $m_{P'} \geqslant 1$ such that $(a, U_{P'}^{(m_{P'})})_{P'} = 0$ and for all $Q \in |\mathfrak{n}|$ with $\pi(Q) = P'$ choose $n_Q \geqslant e(Q/P')m_{P'}$ such that $(\pi^*a, U_Q^{(n_Q)})_Q = 0$. By Lemma 1.4.4 we have

$$\pi_*\left(\bigcap_{Q\in\pi^{-1}(P')} U_Q^{(n_Q)}\right) \subset U_{P'}^{(m_{P'})}.$$

Now by Lemma 1.4.6 we can find a $h \in \mathbb{G}_m(\eta_D)$ such that $h \in U_Q^{(n_Q)}$ for all $Q \in \mathfrak{n} \setminus \pi^{-1}(P)$ and $g/h \in U_Q^{(n_Q)}$ for all $Q \in \pi^{-1}(P)$. We compute:

$$\begin{aligned}(a, \pi_*g)_P &= \sum_{P'\in|\mathfrak{m}|} (a, \pi_*h)_{P'}, && \text{choice of } h,\\ &= -\sum_{P'\in C\setminus|\mathfrak{m}|} (a, \pi_*h)_{P'} && \text{by 1.5.3, (3)},\\ &= -\sum_{Q\in D\setminus|\mathfrak{n}|} \mathrm{Tr}_{x_D/x_C}((\pi^*a, h)_Q) && \text{by (1.18)}\\ &= \sum_{Q\in|\mathfrak{n}|} \mathrm{Tr}_{x_D/x_C}((\pi^*a, h)_Q) && \text{by 1.5.3,(3)},\\ &= \sum_{Q\in\pi^{-1}(P)} \mathrm{Tr}_{x_D/x_C}((\pi^*a, g)_Q) && \text{choice of } h.\end{aligned}$$

Hence (2). □

1.5.6. **Definition.** Let $\mathscr{M}$ be a reciprocity functor, $C \in (\mathscr{C}/S)$ a curve and $P \in C$ a closed point. Then we define

$$\mathrm{Fil}^0_P\mathscr{M}(\eta_C) := \mathscr{M}_{C,P}$$

and for $n \geqslant 1$

$$\mathrm{Fil}^n_P\mathscr{M}(\eta_C) := \{a \in \mathscr{M}(\eta_C) \,|\, (a, u)_P = 0 \text{ for all } u \in U_P^{(n)}\}.$$

Clearly $\mathrm{Fil}^\bullet_P\mathscr{M}(\eta_C)$ forms an increasing and exhaustive filtration of sub-R-modules of $\mathscr{M}(\eta_C)$.

1.5.7. **Definition.** For $n \geqslant 0$ we define $\mathbf{RF}_n$ to be the full subcategory of $\mathbf{RF}$ formed by the reciprocity functors $\mathscr{M}$, which have the property that for any curve $C \in (\mathscr{C}/S)$ and any closed point P in C

$$\mathrm{Fil}^n_P M(\eta_C) = M(\eta_C).$$

We get an increasing sequence of subcategories

$$\mathbf{RF}_0 \subset \mathbf{RF}_1 \subset \ldots \subset \mathbf{RF}_n \subset \ldots \mathbf{RF}.$$

1.5.8. *Remark.* We will see later that the above filtration of $\mathbf{RF}$ is not exhaustive, e.g. $\mathbb{G}_a$ is a reciprocity functor which does not lie in any $\mathbf{RF}_n$, $n \geqslant 0$ (since the pole order is not bounded).

1.5.9. **Lemma.** *Let $\mathscr{M}$ be a reciprocity functor and $\mathfrak{m} = \sum_{P\in C} n_P P$ an effective divisor on a curve $C \in (\mathscr{C}/S)$. Then*

$$\mathscr{M}(C,\mathfrak{m}) = \bigcap_{P\in C} \mathrm{Fil}_P^{n_P}\mathscr{M}(\eta_C) = \bigcap_{P\in|\mathfrak{m}|} \mathrm{Fil}_P^{n_P}\mathscr{M}(\eta_C) \cap \mathscr{M}(C\setminus|\mathfrak{m}|).$$

Proof. The second equality is clear and the inclusion $\supset$ for the first equality follows immediately from reciprocity. For the other inclusion take $a \in \mathscr{M}(C,\mathfrak{m})$, $P \in |\mathfrak{m}|$ and $u \in U_P^{(n_P)}$. Take $N_Q \geqslant n_Q$ with $(a, U_Q^{(N_Q)})_Q = 0$ for all $Q \in |\mathfrak{m}|$ and choose $u_P \in \mathbb{G}_m(\eta)$ with $u_P/u \in U_P^{(N_P)}$ and $u_P \in U_Q^{(N_Q)}$ for $Q \in |\mathfrak{m}| \setminus \{P\}$. Then $u_P \equiv 1 \mod \mathfrak{m}$ and

$$(a,u)_P = (a,u_P)_P = \sum_{Q\in|\mathfrak{m}|} (a,u_P)_Q = 0.$$

Thus $a \in \mathrm{Fil}_P^{n_P}\mathscr{M}(\eta)$, which gives the statement. □

1.5.10. **Lemma.** *If $\mathfrak{m} = \sum_{P\in C} n_P[P]$ is an effective divisor on C, we denote by $\mathfrak{m}_n$ the effective divisor $\mathfrak{m}_n = \sum_{P\in C} m_P[P]$, where $m_P = \min\{n_P, n\}$. For $\mathscr{M} \in \mathbf{RF}$ the following conditions are equivalent:*

(1) *$\mathscr{M}$ lies in $\mathbf{RF}_n$.*
(2) *For any $C \in (\mathscr{C}/S)$ and any effective divisor $\mathfrak{m}$ on C, one has $\mathscr{M}(C,\mathfrak{m}) = \mathscr{M}(C,\mathfrak{m}_n)$.*

Proof. This follows directly from Lemma 1.5.9. □

1.5.11. **Example.** Let $\mathscr{M}$ be a reciprocity functor. Then we have

(1) $\mathscr{M} \in \mathbf{RF}_0$ iff $\mathscr{M}(\eta_C) = \mathscr{M}(C)$ for all $C \in (\mathscr{C}/S)$.
(2) $\mathscr{M} \in \mathbf{RF}_1$ iff $s_0^{\mathscr{M}} = s_1^{\mathscr{M}} : \mathscr{M}_{\mathbb{P}^1_x}(\mathbb{A}^1_x) \to M(x)$.

Indeed (1) follows immediately from the definition and (2) from Lemma 1.4.3.

2. Examples

2.1. **Constant reciprocity functors.** Let M be an R-module. Then M defines a constant Nisnevich sheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$, defined by $M(X) := M^{\oplus r}$, for $X \in \mathrm{Reg}^{\leqslant 1}$ with r connected components, and an elementary correspondence $Z \in \mathrm{Cor}(X,Y)$ acts via multiplication with its degree $\deg(Z/X)$. One easily checks that in this way M defines a reciprocity functor and that we obtain a fully faithful functor

$$(R-\mathrm{mod}) \to \mathbf{RF}_0 \subset \mathbf{RF}.$$

2.2. **Algebraic groups.** In the following by an *algebraic group* we mean a smooth connected and commutative group scheme over S.

2.2.1. *Trace for algebraic groups.* Let G be an algebraic group and $\pi : Y \to X$ a finite and flat morphism of degree d between noetherian S-schemes. Then (see e.g. [1, Exp. XVII, (6.3.4.2)]) there exists a canonical X-morphism

$$X \to \mathrm{Sym}^d_X Y. \tag{2.1}$$

(We quickly recall how this is constructed locally: Assume π corresponds to a ring map $A \to B$ which makes B a free A-module of rank d. For a not necessarily commutative A-algebra D denote by $\mathrm{TS}^d_A(D) = (D^{\otimes_A d})^{\Sigma_d}$ the A-algebra of symmetric tensors. Then (see e.g. [1, Exp. XVII, (6.3.1.4)]) the determinant $\mathrm{End}_A(B) \to A$ induces an homomorphism of A-algebras $\mathrm{TS}^d_A(\mathrm{End}_A(B)) \to A$, whose composition with $\mathrm{End}_A(B) \to \mathrm{TS}^d_A(\mathrm{End}_A(B))$, $x \mapsto x \otimes \ldots \otimes x$, gives back the determinant. Then (2.1) is locally given by taking Spec of the composition

$$\mathrm{TS}^d_A(B) \to \mathrm{TS}^d_A(\mathrm{End}_A(B)) \to A,$$

where the first map is induced by the natural map $B \to \mathrm{End}_A(B)$.)

Now in [1, Exp. XVII, (6.3.13.2)] the trace morphism

$$\pi_* = \mathrm{Tr}_{Y/X} : G(Y) \to G(X)$$

is defined as follows: Let $u : Y \to G$ be an S-morphism, then $\mathrm{Tr}_{Y/X}(u) : X \to G$ is given by the composition

$$X \xrightarrow{(2.1)} \mathrm{Sym}^d_X Y \xrightarrow{\sum_{1=1}^d u} G.$$

By [1, Exp. XVII, Ex. 6.3.18] the trace thus defined equals the usual trace $\Gamma(Y, \mathscr{O}_Y) \xrightarrow{\mathrm{Tr}_{Y/X}} \Gamma(X, \mathscr{O}_X)$ (resp. norm $\Gamma(Y, \mathscr{O}_Y^\times) \xrightarrow{\mathrm{Nm}_{Y/X}} \Gamma(X, \mathscr{O}_X^\times)$) for $G = \mathbb{G}_a$ (resp. $G = \mathbb{G}_m$).

2.2.2. **Proposition** (cf. [31, III])**.** *Let G be an algebraic group. Then the Nisnevich sheaf* $\mathrm{Reg}^{\leqslant 1} \ni U \mapsto G(U)$ *extends to a functor on* $\mathrm{Reg}^{\leqslant 1}\mathrm{Cor}$ *(via the trace recalled above) and as such is a $\mathbb{Z}$-reciprocity functor. Furthermore a morphism of algebraic groups induces a morphism of the corresponding reciprocity functors.*

Proof. It is well-known that $U \to G(U)$ is a Nisnevich sheaf on $\mathrm{Reg}^{\leqslant 1}$. Further, if $U \in \mathrm{RegCon}^{\leqslant 1}$ is 1-dimensional with generic point η the natural map $G(U) \to G(\eta)$ is injective. (This follows from the valuative criterion for properness, since the neutral section $e_G : S \to G$ is a proper map.) Thus the sheaf G satisfies the conditions (Inj) and (P.F.) from Definition 1.3.5. Furthermore by [1, Exp. XVII, Prop. 6.3.15] the trace recalled above yields a functor $G : \mathrm{RegCon}^{\leqslant 1}_* \to (\mathbb{Z} - \mathrm{mod})$, such that for a finite and surjective morphism $\pi : X \to Y$ in $\mathrm{RegCon}^{\leqslant 1}$ the composition $\pi_*\pi^*$ is multiplication with $\deg \pi$ on $G(Y)$. Thus G is a Mackey functor with specialization map if it satisfies condition (3) of Lemma 1.2.2. So let Y, X, X' and f, g be as in Lemma 1.2.2, (3). Since G satisfies (Inj) we may assume that X' actually is an S-point (Then $g : X' \to X$ is either dominant or the inclusion of a closed point.) By [1, Exp. XVII, Prop. 6.3.15] the trace is compatible with base change and decomposes as a sum on disjoint schemes. Hence it suffices to show, that if k is a field and A is a finite local k-algebra, then $\mathrm{Tr}_{A/k} : G(A) \to G(k)$ equals the composition

$$G(A) \to G(A_{\mathrm{red}}) \xrightarrow{\lg(A)\cdot \mathrm{Tr}_{A_{\mathrm{red}}/k}} G(k).$$

This follows immediately from [1, Exp. XVII, Prop. 6.3.5], which implies that if $d_0 = [A_{\mathrm{red}} : k]$ and $d = [A : k] = d_0 \cdot \lg(A)$, then the map $\mathrm{Spec}\, k \to \mathrm{Sym}^d_k(\mathrm{Spec}\, A)$ factors via

$$\mathrm{Spec}\, k \to (\mathrm{Sym}^{d_0}_k(\mathrm{Spec}\, A_{\mathrm{red}}))^{\times \lg(A)} \xrightarrow{\text{can.}} \mathrm{Sym}^d_k(\mathrm{Spec}\, A),$$

where the arrow labeld "can." is the canonical morphism induced by the composition

$$\begin{aligned}\mathrm{TS}^d_k(A) \to \mathrm{TS}^d_k(A_{\mathrm{red}}) &\xrightarrow{\mathrm{TS(diagonal)}} \mathrm{TS}^d_k(A_{\mathrm{red}}^{\oplus \lg(A)}))\\ &\cong \bigoplus_{\sum n_i = d} \mathrm{TS}^{n_1}_k(A_{\mathrm{red}}) \otimes_k \ldots \otimes_k \mathrm{TS}^{n_{\lg(A)}}_k(A_{\mathrm{red}}) \xrightarrow{\text{proj.}} \mathrm{TS}^{d_0}_k(A_{\mathrm{red}})^{\otimes \lg(A)}.\end{aligned}$$

Thus G is a Mackey functor with specialization map. It remains to check that for $C \in (\mathscr{C}/S)$ and any non-empty open subset $U \subset C$ any section $a \in G(U)$ admits a modulus (see Definition 1.4.1). In case $\kappa(x_C)$ is algebraically closed this is a theorem due to M. Rosenlicht (see [31, III, §1, Thm. 1]). The general case follows from this as follows: Let $\sigma : \bar{x} \to x := x_C$ be a geometric point (i.e. $\kappa(\bar{x})$ is algebraically closed), denote by $\nu : D \to (C \times_x \bar{x})_{\mathrm{red}}$ the normalization map (notice that $(C \times_x \bar{x})_{\mathrm{red}}$ is integral since $C \to x$ is geometrically connected) and by π the composition

$$\pi : D \xrightarrow{\nu} (C \times_x \bar{x})_{\mathrm{red}} \subset C \times_x \bar{x} \xrightarrow{\mathrm{id} \times \sigma} C.$$

Take $a \in G(U)$. Then by the theorem of M. Rosenlicht $\pi^* a \in G(\pi^{-1}(U))$ has a modulus and thus we find an effective divisor $\mathfrak{m}$ on C with support equal to $C \setminus U$ and such that $\pi^* a$ has modulus $\pi^* \mathfrak{m}$. We claim that a has modulus $\mathfrak{m}$. For this take $f \in \mathbb{G}_m(\eta_C)$ with $f \equiv 1 \bmod \mathfrak{m}$ (in particular $\pi^* f \equiv 1 \bmod \pi^* \mathfrak{m}$). Since $\sigma : G(x) \to G(\bar{x})$ is injective (G being an fppf-sheaf) it suffices to show

$$0 = \sum_{P \in C \setminus |\mathfrak{m}|} v_P(f) \sigma^* \mathrm{Tr}_{P/x}(s_P(a)) = \sum_{P \in C \setminus |\mathfrak{m}|} \sum_{P' \in P \times_x \bar{x}} v_P(f) \cdot l_{P'} \cdot \sigma^*_{P'} s_P(a),$$

where $\sigma_{P'} : P' \to P$ is induced by σ and $l_{P'} = \lg(\mathscr{O}_{P \times_x \bar{x}, P'})$; notice that we can drop the $\mathrm{Tr}_{P'/\bar{x}}$ on the right since $P' \cong \bar{x}$. For this, let $t \in \mathscr{O}_{C,P}$ be a local parameter, then by [7, Lem. 1.7.2, Ex. 1.2.3] we have

$$l_{P'} \cdot [P'] = [\mathrm{div}(t_{|C \times_x \bar{x}})]_{|P'} = l_{\eta_D} \cdot \sum_{Q \in \nu^{-1}(P')} e(Q/P)) \cdot [P'],$$

where $l_{\eta_D} := \lg(\mathscr{O}_{C \times_x \bar{x}, \eta_D})$ (notice that $[Q : P'] = 1$). Thus we obtain

$$\sum_{P \in C \setminus |\mathfrak{m}|} \sum_{P' \in P \times_x \bar{x}} v_P(f) \cdot l_{P'} \cdot \sigma^*_{P'} s_P(a) = l_{\eta_D} \sum_{Q \in D \setminus |\pi^* \mathfrak{m}|} v_Q(\pi^* f) s_Q(\pi^* a),$$

which is zero by our choice of $\mathfrak{m}$. This finishes the proof. □

2.2.3. *Remark.* It follows from the proof of [31, III, §1, Theorem 1], the above proof and Example 1.5.11 that

$$G \in \begin{cases} \mathbf{RF}_0, & \text{if } G \text{ is an Abelian variety} \\ \mathbf{RF}_1, & \text{if } G \text{ is a semi-Abelian variety} \\ \mathbf{RF} \setminus \cup_n \mathbf{RF}_{n \geqslant 0}, & \text{if } G \text{ is unipotent.} \end{cases}$$

2.3. **Homotopy invariant Nisnevich sheaves with transfers.**

2.3.1. *Recollections on PST.* Let $\mathrm{SmCor_S}$ be the category defined in [35, 2.1] whose objects are smooth S-schemes and the morphisms are given by finite correspondences, i.e. $\mathrm{Hom}_{\mathrm{SmCor_S}}(X, Y) = \mathrm{Cor}_S(X, Y)$. A *presheaf with transfers over S* is a contravariant and additive functor from $\mathrm{SmCor_S}$ to the category of Abelian groups, $\mathscr{F} : (\mathrm{SmCor_S})^{\mathrm{op}} \to (\mathrm{Ab})$. The category of presheaves with transfers is denoted by **PST**; it is an Abelian category. Inside **PST** we have the full subcategories **NST** of Nisnevich sheaves with transfers, **HI** of homotopy invariant presheaves

with transfers and $\mathbf{HI}_{\mathrm{Nis}} = \mathbf{HI} \cap \mathbf{NST}$ of homotopy invariant Nisnevish sheaves with transfers. The inclusion functor $\mathbf{HI} \to \mathbf{PST}$ admits a left adjoint

$$h_0 : \mathbf{PST} \to \mathbf{HI}$$

where $h_0(\mathscr{F})$ is the presheaves with transfers defined for any smooth S-scheme X by

$$h_0(\mathscr{F})(X) := \operatorname{Coker}\left[\mathscr{F}(\mathbb{A}^1_X) \xrightarrow{s_0^* - s_1^*} \mathscr{F}(X)\right].$$

By [35, Theorem 3.1.4] the sheafification functor $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{Nis}}$ restricts to a functor $\mathbf{PST} \to \mathbf{NST}$ which is the left adjoint to the inclusion functor $\mathbf{NST} \to \mathbf{PST}$.

2.3.2. **Definition.** A *model* of $X \in \mathrm{Reg}^{\leqslant 1}$ is a separated and smooth S-scheme U together with an affine morphism $u : X \to U$ such that for any open affine $\operatorname{Spec} A \subset U$ the ring $\Gamma(\operatorname{Spec} A, u_* \mathscr{O}_X)$ is a localization of A by a multiplicative subset. A morphism between two models is a morphism of schemes under X. We denote by $\mathfrak{M}_X$ the category of all models of X.

2.3.3. **Lemma.** *The category $\mathfrak{M}_X$ of models of $X \in \mathrm{Reg}^{\leqslant 1}$ is cofiltered and in particular non-empty.*

Proof. This is straightforward. □

2.3.4. **Proposition.** *Let $\mathscr{F} \in \mathbf{PST}$ be a presheaf with transfers. Then*

$$\mathrm{Reg}^{\leqslant 1} \ni X \mapsto \hat{\mathscr{F}}(X) := \operatorname*{colim}_{U \in \mathfrak{M}_X^{\mathrm{op}}} \mathscr{F}(U)$$

naturally defines a presheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$ (in the sense of Definition 1.2.1, with $R = \mathbb{Z}$) and we obtain an exact functor

$$\mathbf{PST} \to \mathbf{PT}, \quad \mathscr{F} \mapsto \hat{\mathscr{F}},$$

which restricts to $\mathbf{NST} \to \mathbf{NT}$.

Proof. For $X, Y \in \mathrm{Reg}^{\leqslant 1}$ and $[V] \in \mathrm{Cor}(X, Y)$ an elementary correspondence, we find models $X \to U$ and $Y \to V$ and an elementary correspondence $[W] \in \mathrm{Cor}(U, V)$ which pulls back to $[V]$ under $X \times Y \to U \times V$. We get a map $\mathscr{F}([W]) : \mathscr{F}(V) \to \mathscr{F}(U)$ and it is straightforward to see that it induces a map $\hat{\mathscr{F}}(Y) \to \hat{\mathscr{F}}(X)$ which is independent of the choices of U, V, W and is compatible with composition. This proves the first statement and the other statements are immediate. □

The following statement is a collection of results of V. Voevodsky.

2.3.5. **Proposition.** *The above functor $\mathbf{PST} \to \mathbf{PT}$ restricts to a conservative functor*

$$\mathbf{HI}_{\mathrm{Nis}} \to \mathbf{RF}_1, \quad \mathscr{F} \mapsto \hat{\mathscr{F}},$$

where $\mathbf{RF}_1$ is defined in 1.5.7. Furthermore, restricting $\hat{\mathscr{F}}$ to $\mathrm{ptCor}^{\mathrm{op}}$ we obtain a functor $\mathbf{HI}_{\mathrm{Nis}} \to \mathbf{MF}$, which still is conservative.

Proof. Take $\mathscr{F} \in \mathbf{HI}_{\mathrm{Nis}}$. By Proposition 2.3.4 $\hat{\mathscr{F}}$ is a Nisnevich sheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$ and it automatically satisfies condition (F.P.) from Definition 1.3.5. Further, by [34, Corollary 4.19] and [35, Proposition 3.1.11], the restriction map $\mathscr{F}(X) \hookrightarrow \mathscr{F}(U)$, for $U \hookrightarrow X$ a dense open immersion in Sm_S, is injective. This implies (Inj) from Definition 1.3.5 and hence $\hat{\mathscr{F}} \in \mathbf{MFsp}$. Furthermore, since $\mathscr{F}$ is homotopy invariant we have $i_0^* = i_\infty^*$ on $\hat{\mathscr{F}}(\mathbb{P}^1_x \setminus \{1\})$ with x an S-point. Thus $\hat{\mathscr{F}} \in \mathbf{RF}_1$ by Lemma 1.4.3. The conservativity statement follows immediately from [34, Proposition 4.20] and [35, Proposition 3.1.11]. □

2.4. **Cycle modules.**

2.4.1. In the following we will freely use the numbering of the data and rules a cycle module should satisfy as introduced by M. Rost in [27]. We will check that a cycle module defines a reciprocity functor in the sense of Definition 1.5.1. For this we will use Proposition 1.3.6, which provides a description of Mackey functor with specialization which is closer to the definition of cycle modules.

2.4.2. *Cycle modules are graded reciprocity functors.* Let M_* be a cycle module as defined in [27, (2.1) Definition]. Let $n \in \mathbb{Z}$ be an integer. First note that M_n is a Mackey functor by **R1a-c** and **R2d**, see Remark 1.3.3. As we shall see, the Mackey functor M_n is canonically the underlying Mackey functor of a Nisnevich sheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$ which satisfy the modulus condition (MC). Let $C \in \mathscr{C}/S$ be a curve. To a closed point P in C corresponds a discrete valuation v_P on the function field $\kappa(\eta_C)$ of the curve, and thus a residue homomorphism **D4**

$$\partial_P := \partial_{v_P} : M_n(\eta_C) \to M_{n-1}(P).$$

Given an open subset $U \subseteq C$, we set

$$(\mathscr{M}_n)_C(U) := \bigcap_{P \in U} \ker\left(\partial_P : M_n(\eta_C) \to M_{n-1}(P)\right). \tag{2.2}$$

This in fact defines a sheaf in the Zariski topology on C and the stalk at P is the kernel of ∂_P.

2.4.3. **Lemma.** *For all $C \in (\mathscr{C}/S)$ the functor on the small Nisnevich site C_{Nis}, which sends an étale C-scheme U to $\oplus_i(\mathscr{M}_n)_{C_i}(U_i)$, where the U_i's are the connected components of U with projective regular model C_i, is a sheaf.*

Proof. Let $X \in \mathrm{Reg}^{\leqslant 1}$ and

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d, "\pi"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

be a distinguished Nisnevich square for the Nisnevich topology *i.e.* π is an étale morphism, j an open immersion and if $Z := X \backslash U$ with its reduced scheme structure, then $\pi^{-1}(Z) \to Z$ is an isomorphism. By [38, Proposition 2.17], it is enough to check that the square

$$\begin{tikzcd} \mathscr{M}_n(X) \arrow[r] \arrow[d] & \mathscr{M}_n(V) \arrow[d] \\ \mathscr{M}_n(U) \arrow[r] & \mathscr{M}_n(W) \end{tikzcd} \tag{2.3}$$

is cartesian. Let for any scheme $Y \in \mathrm{Reg}^{\leqslant 1}$, $\mathscr{N}_n(Y)$ be the direct sum over the irreducible components T of Y of the $M_n(\eta_T)$'s. Then the square analogous to (2.3) with $\mathscr{M}_n$ replaced by $\mathscr{N}_n$ is cartesian. Indeed this follows from the following two observations:

- if C is an irreducible component of X, then either $C \cap U \neq \emptyset$ (and thus $C \cap U$ is an irreducible component of U), or C is contained in Z, and therefore dominated by a unique irreducible component of V, which is moreover isomorphic to C;
- since π is étale, any irreducible component D of V dominates an irreducible component C of X, and $C \cap U \neq \emptyset$ if and only if $D \cap W \neq \emptyset$.

Now to see that (2.3) is cartesian, it is enough to remark that if $P \in X$ is a closed point that does not lie in U, then there exists $Q \in V$ such that π induces an isomorphism $\kappa(Q) \simeq \kappa(P)$, then by **R3a** we have

$$\partial_Q \circ \pi^* = e(Q/P) \cdot \partial_P = \partial_P$$

since π is unramified. □

2.4.4. Let ϖ be a uniformizer of the local ring $\mathscr{O}_{C,P}$, and consider as in [27, p. 329], the map

$$s_P^{\varpi} : M_n(\eta_C) \to M_n(P), \quad a \mapsto s_P^{\varpi}(a) := \partial_P(\{-\varpi\} \cdot a).$$

The restriction to $(\mathscr{M}_n)_{C,P}$ is independent of the choice of the uniformizer ϖ. Indeed if ϖ' is another uniformizer of the local ring $\mathscr{O}_{C,P}$, then there exists a unit $u \in \mathscr{O}_{C,P}^{\times}$ such that $\varpi' = u\varpi$ and using **R3e**, we get for any $a \in (\mathscr{M}_n)_{C,P}$

$$\begin{aligned} s_P^{\varpi'}(a) &= \partial_P(\{-\varpi'\} \cdot a) = \partial_P(\{-u\varpi\} \cdot a) = \partial_P(\{-\varpi\} \cdot a) + \partial_P(\{u\} \cdot a) \\ &= s_P^{\varpi}(a) - \{\bar{u}\} \cdot \partial_P(a) = s_P^{\varpi}(a). \end{aligned}$$

Hence we may denote by

$$s_P : (\mathscr{M}_n)_{C,P} \to M_n(P)$$

the restriction of the map s_P^{ϖ}.

2.4.5. **Proposition.** *Let M_* be a cycle module over S. Then for any integer $n \in \mathbb{Z}$, the triple $(M_n, \mathscr{M}_n, s)$ defines a $\mathbb{Z}$-reciprocity functor over S via Proposition 1.3.6, which lies in* $\mathbf{RF}_1$. *Furthermore, its associated symbol (see Corollary-Definition 1.5.3) is given by*

$$M_n(\eta_C) \times \mathbb{G}_m(\eta_C) \to M_n(x_C), \quad (a, f) \mapsto (a, f)_P = \mathrm{Tr}_{P/x_C} \partial_P(\{f\} \cdot a), \tag{2.4}$$

for $C \in (\mathscr{C}/S)$ and $P \in C$.

Proof. Let us first check that $\mathscr{M}_n$ has the required functoriality. Let $\pi : D \to C$ be a morphism in $(\mathscr{C}/S)$, then the morphism $\pi^* : M_n(\eta_C) \to M_n(\eta_D)$ induces a morphism of presheaves $\pi^* : (\mathscr{M}_n)_C \to \pi_*(\mathscr{M}_n)_D$. Indeed if $U \subseteq C$ is an open subset, then by **R3a** we have

$$\partial_Q \circ \pi^* = e(Q/P) \cdot (\pi_Q^* \circ \partial_P)$$

for any closed points $P \in U$ and $Q \in D$ such that $\pi(Q) = P$. Hence if $a \in (\mathscr{M}_n)_C(U)$ then $\pi^* a$ lies in $(\mathscr{M}_n)_D(\pi^{-1}(U))$. Similarly, if π is a finite morphism, **R3b** ensures that the morphism $\pi_* : M_n(\eta_D) \to M_n(\eta_C)$ induces a morphism of presheaves $\pi_* : \pi_*(\mathscr{M}_n)_D \to (\mathscr{M}_n)_C$. Thus the triple $(M_n, \mathscr{M}_n, s)$ is a data (1)-(3) from Proposition 1.3.6. It remains to check the conditions (R0)-(S3) from Proposition 1.3.6 and (MC) from Definition 1.4.1.

Condition (RS0) is Lemma 2.4.3 and (RS1) is an immediate consequence of the condition **(FD)** that cycle modules are required to fulfill. Let $C \in (\mathscr{C}/S)$ and $a \in M_n(x_C)$. Then by **R3c**, we have $\partial_P(\rho^* a) = 0$ for all $P \in C$, which means that (RS2) holds. The condition (S3) follows from **R3d**. Let $\pi : D \to C$ be a morphism in $(\mathscr{C}/S)$, $Q \in D$ and denote by $\pi_Q : Q \to \pi(Q) =: P$ the map induced by π. Let ϖ_Q (resp. ϖ_P) be a uniformizer of the local ring $\mathscr{O}_{D,Q}$ (resp. $\mathscr{O}_{C,P}$). We may write

$$\pi^* \varpi_P = u \cdot \varpi_Q^{e(Q/P)},$$

where $u \in \mathscr{O}_{D,Q}^{\times}$. Relation (S1) follows from [27, (1.9) Lemma], which gives for any $a \in (\mathscr{M}_n)_{C,P}$

$$s_Q^{\varpi_Q}(\pi^* a) = \pi_Q^* s_P^{\varpi_P}(a) - \{\bar{u}\} \cdot \pi_Q^* \partial_P(a) = \pi_Q^* s_P(a).$$

Assume now that π is finite. For any $a \in \pi_*((\mathscr{M}_n)_D)_P$, we have

$$\begin{aligned} s_P(\pi_* a) &= \partial_P(\{-\varpi_P\} \cdot \pi_* a) = \partial_P \pi_* \left(\{-\pi^* \varpi_P\} \cdot a\right) && \text{by } \mathbf{R2b} \\ &= \sum_{Q \in \pi^{-1}(P)} \pi_{Q*} \partial_Q \left(\{-\pi^* \varpi_P\} \cdot a\right) && \text{by } \mathbf{R3b} \end{aligned}$$

On the other hand by **R3e**

$$\begin{aligned} \partial_Q \left(\{-\pi^* \varpi_P\} \cdot a\right) &= e(Q/P) \cdot \partial_Q(\{-\varpi_Q\} \cdot a) + \partial_Q(\{u\} \cdot a) \\ &= e(Q/P) \cdot s_Q(a) - \{\bar{u}\} \cdot \partial_Q(a) \\ &= e(Q/P) \cdot s_Q(a). \end{aligned}$$

This implies (S2). It remains to prove that the modulus condition (MC) is fulfilled. Let $C \in (\mathscr{C}/S)$ be a curve and $U \subset C$ be a non-empty open subset and take $a \in (\mathscr{M}_n)_C(U)$. The closed subset $C \setminus U$ consists then of finitely many closed points $P_1, \ldots, P_r$ and let $\mathfrak{m}$ be the effective divisor $\mathfrak{m} := \sum_{i=1}^r P_i$. Let $f \in \mathbb{G}_m(\eta_C)$ be a rational function such that $f \equiv 1 \mod \mathfrak{m}$. By the reciprocity law for curves **RC**, we know that

$$\sum_{P \in C} \mathrm{Tr}_{P/x_C} \partial_P(\{f\} \cdot a) = 0. \tag{2.5}$$

If $P = P_i$ for some $i \in \{1, \ldots, r\}$, then by definition $f \in U_P^{(1)} = 1 + \mathfrak{m}_P$. Hence f is a unit and **R3e** implies that

$$\partial_P(\{f\} \cdot a) = -\{\bar{f}\} \cdot \partial_P(a) = -\{1\} \cdot \partial_P(a) = 0. \tag{2.6}$$

On the other hand, if $P \in U$ and ϖ is a uniformizer of the local ring $\mathscr{O}_{C,P}$, then by **R3f** we have

$$\begin{aligned} \partial_P(\{f\} \cdot a) &= \partial_P(\{f\}) \cdot s_P^{\varpi}(a) - s_P^{\varpi}(f) \cdot \partial_P(a) + \{-1\} \cdot \partial_P(\{f\}) \cdot \partial_P(a) \\ &= \partial_P(\{f\}) \cdot s_P^{\varpi}(a) = v_P(f) \cdot s_P(a) \end{aligned}$$

since $a \in (\mathscr{M}_n)_C(U)$. Hence the reciprocity law (2.5), may be rewritten as

$$\sum_{P \in C \setminus |\mathfrak{m}|} v_P(f) \cdot \mathrm{Tr}_{P/x_C}(s_P(a)) = 0.$$

This shows that the modulus condition (MC) is fulfilled. Hence $(M_n, \mathscr{M}_n, s)$ defines a reciprocity functor. The formula for the associated local symbol follows from the computation above and the uniqueness statement in Corollary-Definition 1.5.3. Hence $(M_n, \mathscr{M}_n, s) \in \mathbf{RF}_1$ by (2.6). $\square$

2.4.6. **Example.** In particular the n-th level of Milnor K-theory gives a reciprocity functor for all integers $n \geqslant 0$,

$$\mathrm{K}_n^{\mathrm{M}} = (\mathrm{K}_n^{\mathrm{M}}, \mathscr{K}_n^{\mathrm{M}}, s).$$

Notice that for $C \in (\mathscr{C}/S)$, the sheaf $\mathscr{K}_{n,C}^{\mathrm{M}}$ is defined by (2.2). If $\kappa(x_C)$ is an infinite field this coincides with the sheaf $\mathrm{K}_n^{\mathrm{M}}(\mathscr{O}_C)$ by [19, Thm. 7.1.].

Also notice that we have

$$\mathrm{K}_1^{\mathrm{M}} = \mathbb{G}_m.$$

2.4.7. *Functors of zero cycles.* Let X be a smooth quasi-projective S-scheme, of pure dimension d, and $n \in \mathbb{Z}$ an integer. Denote by $\mathrm{CH}_0(X, n) = \mathrm{CH}^{d+n}(X, n)$ the higher Chow group of zero cycles defined by S. Bloch. As explained in Proposition 2.4.5, cycle modules are graded reciprocity functors. This implies in particular that the functor

$$\mathrm{CH}_0(X, n) : (\mathrm{pt}/S) \to (\mathbb{Z} - \mathrm{mod}), \quad x \mapsto \mathrm{CH}_0(X, n)(x) := \mathrm{CH}_0(X_x, n), \tag{2.7}$$

defines a $\mathbb{Z}$-reciprocity functor over S, denoted by

$$\mathscr{C}H_0(X,n).$$

Indeed this functor may be obtained from $\mathrm{K}^{\mathrm{M}}_*$ via the fibration technics of [27, §7]. To see this let M_* be a cycle module over X (e.g. Milnor K-theory $\mathrm{K}^{\mathrm{M}}_*$) and $\varrho : X \to S$ be the structural morphism. As shown by M. Rost in [27, (7.3) Theorem], for any integer $q \in \mathbb{Z}$, we get a cycle module $A_q[\varrho; M]$ over S. This cycle module is such that for any S-point x

$$A_q[\varrho; M]_n(x) = A_q(X_x; M, n),$$

where $A_q(X_x; M, n)$ is the q-th homology of the cycle complex $C_*(X_x; M, n)$ defined in [27, §5]. Assume now $M_* = \mathrm{K}^{\mathrm{M}}_*$. We have then two isomorphisms

$$A_0[\varrho; \mathrm{K}^{\mathrm{M}}_*]_n(x) = A_0(X_x; \mathrm{K}^{\mathrm{M}}_*, n) \simeq \mathrm{H}^d_{\mathrm{Zar}}(X_x; \mathscr{K}^{\mathrm{M}}_{d+n}) \simeq \mathrm{CH}_0(X_x, n). \tag{2.8}$$

The first one is due to K. Kato [17] (see also [27, (6.5) Corollary]), while the second one may be found in [2, Theorem 5.5]. Since $A_0[\varrho, \mathrm{K}^{\mathrm{M}}_*]$ is a cycle module, it follows from Proposition 2.4.5 that (2.7) defines a reciprocity functor.

2.4.8. *Remark.* Let $C \in (\mathscr{C}/S)$ be a curve and P be a closed point in C. As part of the cycle module structure, we have a residue homomorphism

$$\partial_P : A_0[\rho; \mathrm{K}^{\mathrm{M}}_*]_n(\eta_C) \to A_0[\rho; \mathrm{K}^{\mathrm{M}}_*]_{n-1}(P).$$

For $n = 0$ the right hand side vanishes and so does the residue map ∂_P. Hence it follows from the definition (2.2) of the regular structure on the reciprocity functor $\mathscr{C}H_0(X)$ that for any open subset $U \subseteq C$

$$\mathscr{C}H_0(X)(U) = \mathrm{CH}_0(X_{\eta_C}).$$

In particular, $\mathscr{C}H_0(X) \in \mathbf{RF}_0$. Moreover the specialization map is the one defined *e.g.* in [7, 20.3].

2.4.9. Let X be a smooth and quasi-projective S-scheme of equidimension d and $n \geqslant 0$. For Y, Z two smooth connected schemes and $\gamma \in \mathrm{Cor}(Y, Z)$ a correspondence, define a map

$$\gamma^* : \mathrm{CH}^{d+n}(X \times Z, n) \to \mathrm{CH}^{d+n}(X \times Y, n), \quad \alpha \mapsto p_{12*}(p_{13}^*\alpha \cdot p_{23}^* cl(\gamma)),$$

where p_{12}, p_{13}, p_{23} are the three projections from $X \times Y \times Z$ to $X \times Y$, $X \times Z$, $Y \times Z$ respectively and $cl(\gamma)$ denotes the class of γ in $\mathrm{CH}^{\dim Z}(Y \times Z)$. (Notice that $p_{13}^*\alpha \cdot p_{23}^* cl(\gamma) = p_{23}^* cl(\gamma) \cdot p_{13}^*\alpha$.) One easily checks that this defines a homotopy invariant presheaf with transfers

$$\mathrm{CH}^{d+n}(X, n) : Sm\mathrm{Cor} \to (\mathbb{Z}-\mathrm{mod}), \quad Y \mapsto \mathrm{CH}^{d+n}(X, n)(Y) := \mathrm{CH}^{d+n}(X \times Y, n).$$

We denote by $\mathrm{CH}^{d+n}_{\mathrm{Nis}}(X, n) \in \mathbf{HI}_{\mathrm{Nis}}$ its Nisnevich sheafification. Applying the functor $\widehat{\ } : \mathbf{HI}_{\mathrm{Nis}} \to \mathbf{RF}$ from Proposition 2.3.5 we get a canonical isomorphism of reciprocity functors

$$\mathscr{C}H_0(X, n) \cong \mathrm{CH}^{d+n}_{\mathrm{Nis}}(X, n)\widehat{\ }. \tag{2.9}$$

Indeed by construction we get a canonical isomorphism of Mackey functors (see Definition 1.3.1) and thus by Proposition 1.3.6 we have to check that the specialization maps are the same. But for $C \in (\mathscr{C}/S)$, $U \subset C$ open and $P \in U$ the specialization map $s_P : \mathscr{C}H_0(X, n)(U) \to \mathscr{C}H_0(X, n)(P)$ is defined using the residue homomorphism ∂_P of the underlying Rost's cycle module (see 2.4.4), which in turn is defined using the differentials in the cycle complex $C_*(X \times V; \mathrm{K}^{\mathrm{M}}_*, n)$, where $V \in Sm$ is a model of U, see [27, 7.]. On the other hand the cycle complex $C_*(X \times V; \mathrm{K}^{\mathrm{M}}_*, n)$ is nothing but the Gersten resolution on $X \times V$ of $\mathscr{K}^{\mathrm{M}}_{d+n}$, which is isomorphic to the Gersten resolution of $\mathrm{CH}^{d+n}_{\mathrm{Nis}}(X, n)$ as a Zariski sheaf on $X \times V$ (see e.g. [9, Lemma

3.2]). It is this isomorphism which induces the isomorphism (2.8). But the specialization map for $\mathrm{CH}^{d+n}_{\mathrm{Nis}}(X,n)$ is again induced by the differentials of the Gersten resolution (see [2, Lem. 6.3]), thus we get the compatibility with the specialization maps.

2.5. **Kähler Differentials.**

2.5.1. *Trace for Kähler differentials.* Let T be a scheme, Y a noetherian T-scheme and $f : X \to Y$ a finite morphism, which is a complete intersection, i.e. f is flat and for any $x \in X$ there exists an open neighborhood U of x, such that $f_{|U}$ factors as the composition of a regular closed embedding followed by a smooth morphism, $U \hookrightarrow P \to Y$. (This is for example satisfied if X and Y are regular and f is finite and flat; also any base change with respect to Y will again be a complete intersection.) Then in [20, §16] there is constructed for all q a trace (pushforward)

$$f_* = \mathrm{Tr}_{X/Y} : f_*\Omega^q_{X/T} \to \Omega^q_{Y/T},$$

which has the following properties:

(Tr1) For $\alpha \in \Omega^i_{Y/T}$ and $\beta \in \Omega^{q-i}_{X/T}$, we have $f_*(f^*(\alpha)\beta) = \alpha f_*(\beta)$.

(Tr2) For $q = 0$ the map $f_* : f_*\mathscr{O}_X \to \mathscr{O}_Y$ is the usual trace for a finite and locally free extension of algebras.

(Tr3) If $g : Y' \to Y$ is a T-morphism of noetherian schemes, then the base change $f' : X' = X \times_Y Y' \to Y'$ of f is finite and a complete intersection and the following diagram commutes

$$\begin{tikzcd} f_*\Omega^q_{X/T} \arrow[r, "g'^*"] \arrow[d, "f_*"'] & h_*\Omega^q_{X'/T} \arrow[d, "f'_*"] \\ \Omega^q_{Y/T} \arrow[r, "g^*"] & g_*\Omega^q_{Y'/T}, \end{tikzcd}$$

where $g' : X' \to X$ is the base change of g and $h := f \circ g' = g \circ f'$.

(Tr4) If $f_i : X_i \to Y$, $i = 1, \dots, n$, are finite morphisms, which are local complete intersections, then so is $f := \sqcup f_i : X := \sqcup_{i=1}^n X_i \to Y$ and for $\beta = (\beta_1, \dots, \beta_n) \in f_*\Omega^q_{X/T} = \oplus_i f_{i*}\Omega^q_{X_i/T}$ the following equality holds in $\Omega^q_{Y/T}$

$$f_*(\beta) = \sum_i f_{i*}(\beta_i).$$

(Tr5) If $g : W \to X$ is finite and a complete intersection, then so is $f \circ g : W \to Y$ and we have

$$f_*g_* = (f \circ g)_* : (f \circ g)_*\Omega^q_{Y/T} \to \Omega^q_{S/T}.$$

(Tr6) $f_* \circ d = d \circ f_* : f_*\Omega^q_{X/T} \to \Omega^{q+1}_{Y/T}$.

(Tr7) For all $a \in f_*\mathscr{O}_S^\times$, we have

$$f_*(\frac{da}{a}) = \frac{d\mathrm{Nm}_{X/Y}(a)}{\mathrm{Nm}_{X/Y}(a)},$$

where $\mathrm{Nm}_{X/Y} : f_*\mathscr{O}_X^\times \to \mathscr{O}_Y^\times$ is the norm map.

Furthermore using [20, Thm 16.1] and [20, §16, Exercise 5)] we obtain: Let x be an S-point and $\varphi : Y \to x$ a finite morphism, which is a complete intersection. Then

the following diagram commutes for all $q \geqslant 0$

$$\begin{tikzcd} \Omega^q_{Y/T} \arrow[r] \arrow[d, "\varphi_*"'] & \bigoplus_{y\in Y} \Omega^q_{y/T} \arrow[dl, "\sum_y l_y \varphi_{y*}"] \\ \Omega^q_{x/T}, & \end{tikzcd} \tag{2.10}$$

where φ_y is the composition $y \hookrightarrow Y \xrightarrow{\varphi} x$, $l_y = \mathrm{length}(\mathscr{O}_{Y,y})$ and the horizontal map is induced by the natural surjections $\Omega^q_{Y/T} \to \Omega^q_{y/T}$.

2.5.2. *Residue map for Kähler differentials.* Take C in $(\mathscr{C}/S)$ and set $K = \kappa(C)$, $k = \kappa(x_C)$ and $x = x_C$. For $n \in \mathbb{N}$ set $K_n := k(K^{p^n})$ and viewing K_n and $\mathscr{O}_C$ as subsheaves of the constant sheaf K on C we define $C_n = \mathrm{Spec}(\mathscr{O}_C \cap K_n)$. Then (see [13, p. 88 and Thm 3]) we obtain maps

$$C = C_0 \to C_1 \to C_2 \to \dots,$$

which are homeomorphisms and each C_n is a regular projective and connected curve over x $(= x_{C_n})$. Further, for $P \in C$ denote its image in C_n by P_n. Then, by [13, Thm1, Thm4], for almost all n the curve C_n is smooth over x and $\kappa(P_n)$ is the separable closure of k in $\kappa(P)$.

In [20, §17, Def. 17.4.] the residue map

$$\mathrm{Res}_P : \Omega^1_{\eta/x} \to k$$

is defined by

$$\mathrm{Res}_P(\alpha) := \mathrm{Tr}_{P_n/x}(\mathrm{Res}_t(\mathrm{Tr}_{C/C_n}\alpha)), \quad \alpha \in \Omega^1_{\eta/x},$$

where Tr_{C/C_n} and $\mathrm{Tr}_{P_n/x}$ are the corresponding traces from 2.5.1, n is chosen such that $\kappa(P_n)$ is the separable closure of k in $\kappa(P)$, which implies that the choice of a local parameter t at P_n determines a unique and continuous isomorphism of k-algebras between $\kappa(P_n)((t))$ and the completion of K_n at P_n, and $\mathrm{Res}_t : \Omega^1_{\kappa(P_n)((t))/\kappa(P_n)} \to \kappa(P_n)$ is the usual residue. It is shown in [20, §17], that this definition does not depend on the choices made. Furthermore the following properties are proven:

(1) Res_P is k-linear and factors over $\Omega^1_{\eta/x} \to H^1_P(\Omega^1_{\eta/x}) = \Omega^1_{\eta/x}/\Omega^1_{C/x,P}$.
(2) Let $\pi : D \to C$ be a finite morphism in $(\mathscr{C}/S)$ and $P \in C$, then on $\Omega^1_{\eta_D/x_D}$

$$\mathrm{Res}_P \circ \mathrm{Tr}_{D/C} = \sum_{Q \in \pi^{-1}(P)} \mathrm{Tr}_{x_D/x} \circ \mathrm{Res}_Q$$

(3) For $C \in (\mathscr{C}/S)$ we have for all $\alpha \in \Omega^1_{\eta/x}$

$$\sum_{P\in C} \mathrm{Res}_P(\alpha) = 0.$$

2.5.3. *Remark.* Notice that if $\kappa(P)$ is separable over k, then $\mathrm{Res}_P : \Omega^1_{\eta/x} \to k$, factors via a *fine residue map*

$$\Omega^1_{\eta/x} \xrightarrow{\widetilde{\mathrm{Res}}_P} \kappa(P) \xrightarrow{\mathrm{Tr}} k,$$

where $\widetilde{\mathrm{Res}}_P$ is defined to be Res_t, with t a local parameter at P. This is independent of the choice of t by the same argument as in [31, II, §11].

2.5.4. *Higher residues.* Let $q \geqslant 0$ be a non-negative integer. For any in $\mathrm{Reg}^{\leqslant 1}$ we can consider the q-th absolute Kähler differentials of X over $\operatorname{Spec}\mathbb{Z}$,

$$\Omega^q_X := \Omega^q_{X/\operatorname{Spec}\mathbb{Z}}.$$

For $C \in (\mathscr{C}/S)$ and $P \in C$ we define

$$\mathrm{Res}^{q+1}_P : \Omega^{q+1}_\eta \to \Omega^q_{x_C}$$

as follows: Take $\pi : C \to C'$ a finite, surjective and purely inseparable morphism in $(\mathscr{C}/S)$ such that C' is generically smooth over $x = x_C = x_{C'}$. (Notice that such a π is automatically a homeomorphism.) Let $P' := \pi(P)$ be the image of P in C' and $\eta' = \pi(\eta)$ the generic point of C'. Then define Res^{q+1}_P as the composition

$$\Omega^{q+1}_\eta \xrightarrow{\pi_*} \Omega^{q+1}_{\eta'} \twoheadrightarrow \Omega^1_{\eta'/x} \otimes_{\kappa(x)} \Omega^q_x \xrightarrow{\mathrm{Res}_{P'}\otimes \mathrm{id}} \Omega^q_x,$$

where $\mathrm{Res}_{P'} : \Omega^1_{\eta/x} \to \kappa(x)$ is the residue map from 2.5.2 and the surjection is induced by the short exact sequence

$$0 \to \kappa(\eta') \otimes \Omega^1_{\kappa(x)} \to \Omega^1_{\eta'} \to \Omega^1_{\eta'/x} \to 0.$$

This definition is independent of the chosen finite, surjective and purely inseparable x_C-morphism $\pi : C \to C'$ with C' generically smooth over x_C. Indeed since such a morphism corresponds (up to isomorphism) to giving a finite purely inseparable field extension $\kappa(C)/K'$ such that K' is separable over $\kappa(x_C)$ we only have to show, that if $C' \to C''$ is a finite, surjective and purely inseparable x_C-morphism between generically smooth curves, then the residues constructed with respect to $\pi : C \to C'$ and $\pi' : C \to C' \to C''$ coincide. But since the kernel of the surjection $\Omega^{q+1}_{\eta'} \twoheadrightarrow \Omega^1_{\eta'/x} \otimes_{\kappa(x)} \Omega^q_x$ equals the image of $\kappa(\eta') \otimes_{\kappa(x)} \Omega^{q+1}_x \to \Omega^{q+1}_{\eta'}$ and the same for η'' the linearity of the trace yields a commutative diagram

$$\begin{tikzcd}
\Omega^{q+1}_{\eta'} \arrow[r, two heads] \arrow[d, "\mathrm{Tr}_{\eta'/\eta''}"'] & \Omega^1_{\eta'/x} \otimes_{\kappa(x)} \Omega^q_x \arrow[d, "\mathrm{Tr}_{\eta'/\eta''}\otimes \mathrm{id}"] \\
\Omega^{q+1}_{\eta''} \arrow[r, two heads] & \Omega^1_{\eta''/x} \otimes_{\kappa(x)} \Omega^q_x.
\end{tikzcd}$$

Now it follows from 2.5.2, (2) and the transitivity of the trace that the definition of Res^{q+1}_P does not depend on the choice of π. In particular (see 2.5.2) we can choose $\pi : C \to C'$ in such a way that C' is smooth over x and that $\kappa(P')/\kappa(x)$ is separable, which simplifies the definition of Res_P. It also follows that $\mathrm{Res}^1_P = \mathrm{Res}_P$ and that Res^{q+1}_P satisfies the analog of the properties 2.5.2, (1), (2), (3).

2.5.5. **Theorem.** *For all $q \geqslant 0$ the absolute Kähler differentials define a reciprocity functor*

$$\mathrm{Reg}^{\leqslant 1}\mathrm{Cor} \to (\mathbb{Z} - \mathrm{mod}), \quad X \mapsto \Omega^q_X,$$

such that $\Omega^q([\Gamma_f]) = f^*$ is the usual pullback on Ω^q (resp. $\Omega^q([\Gamma^t_f]) = f_*$ is the trace recalled in 2.5.1) for $f : X \to Y$ a map in $\mathrm{Reg}^{\leqslant 1}$ (resp. a finite and flat map in $\mathrm{Reg}^{\leqslant 1}$). Furthermore the local symbol attached to Ω^q (see Corollary-Definition 1.5.3) is given by

$$(\alpha, f)_P = \mathrm{Res}^{q+1}_P(\tfrac{df}{f}\alpha), \quad \alpha \in \Omega^q_{\eta_C}, f \in \mathbb{G}_m(\eta_C)$$

for all $C \in (\mathscr{C}/S)$ and $P \in C$.

Proof. It follows from Lemma 1.2.2 and (Tr1)-(Tr5) and (2.10) in 2.5.1, that Ω^q is a presheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$ with pullback and pushforward as in the statement. Further it clearly is a Nisnevich sheaf and satisfies (F.P.) from Definition

1.3.5. Next observe that for $C \in (\mathscr{C}/S)$ and $P \in C$ the module $\Omega^q_{C,P}$ is free (of maybe infinite rank). Indeed if C/x_C is smooth we have the exact sequence

$$0 \to \mathscr{O}_{C,P} \otimes_{\kappa(x_C)} \Omega^1_{x_C} \to \Omega^1_{C,P} \to \Omega^1_{C/x_C,P} \to 0,$$

which implies that $\Omega^1_{C,P}$ and hence also all its exterior powers are free. In case $\kappa(x_C)$ has positive characteristic and C is only regular this is e.g. [20, Thm. 7.5]. In particular $\Omega^q_{C,P}$ is a submodule of Ω^q_η and hence property (Inj) from Definition 1.3.5 holds. Thus $\Omega^q \in \mathbf{MFsp}$ and it remains to check condition (MC). For this we claim that for $C \in (\mathscr{C}/S)$, $P \in C$, $\alpha \in \Omega^q_{C,P}$ and t a local parameter at P, we have (with $x := x_C$ and the notation from 1.3.2)

$$\mathrm{Tr}_{P/x}(s_P(\alpha)) = \mathrm{Res}^{q+1}_P(\tfrac{dt}{t}\alpha). \tag{2.11}$$

Then $\mathbb{G}_m(\eta_C) \to \Omega^q_x$, $f \mapsto \mathrm{Res}^{q+1}_P(\frac{df}{f}\alpha)$ clearly is a group homomorphism, which is continuous (by 2.5.2, 1.) and satisfies the conditions (b) and (c) from Proposition 1.4.7. Hence Ω^q satisfies (MC) and is a reciprocity functor and the explicit description for the local symbol given in the theorem follows from the uniqueness statement in Proposition 1.4.7. Thus it suffices to prove the claim (2.11). In case C/x is smooth and P/x is separable we have that $\mathscr{O}_{C,P}$ is étale over $\kappa(x)[t]$. Thus

$$\Omega^q_{C,P} = (\mathscr{O}_{C,P} \otimes \Omega^q_{\kappa(x)}) \oplus (\mathscr{O}_{C,P} \otimes \Omega^{q-1}_{\kappa(x)} \cdot dt).$$

Therefore we can write $\alpha = \alpha_0 + \alpha_1 dt$ with $\alpha_i \in \mathscr{O}_{C,P} \otimes \Omega^{q-i}_{\kappa(x)}$ and we have by definition

$$\mathrm{Res}^{q+1}_P(\tfrac{dt}{t}\alpha) = \mathrm{Tr}_{P/x}(s_P(\alpha_0)) = \mathrm{Tr}_{P/x}(s_P(\alpha)).$$

The general case thus follows from the analog of 2.5.2, 1. and 2. for Res^{q+1}_P and the following lemma. □

2.5.6. **Lemma.** *Let $\pi : D \to C$ be a finite, surjective and purely inseparable morphism between regular curves over a field k of characteristic $p > 0$. Let $Q \in D$ be a closed point and $P = \pi(Q)$ its image in C and let $z \in \mathscr{O}_{D,Q}$ and $t \in \mathscr{O}_{C,P}$ be local parameters at Q and P respectively. Then for all $\alpha \in \Omega^q_{D,Q}$ there exists a $\beta \in \Omega^q_{C,P}$ such that*

$$\mathrm{Tr}_{D/C}(\tfrac{dz}{z}\alpha) \equiv \tfrac{dt}{t}\beta \;\textit{mod}\; \Omega^{q+1}_{C,P} \quad \textit{and} \quad s_P(\beta) = \mathrm{Tr}_{Q/P}(s_Q(\alpha)) \quad \textit{in}\; \Omega^q_P.$$

Proof. Notice first that $\operatorname{Spec}\mathscr{O}_{D,Q} \to \operatorname{Spec}\mathscr{O}_{C,P}$ is a complete intersection morphism and a homeomorphism. Thus $\mathrm{Tr}_{C/D}$ maps $\Omega^{q+1}_{D,Q}$ to $\Omega^{q+1}_{C,P}$ (by (Tr3)). Hence it suffices to prove the statement in the case $[D : C] = p = e(Q/P) \cdot f(P/Q)$.

1. case: $e(Q/P) = 1$, $f(Q/P) = p$. We can write $t = zu$ with $u \in \mathscr{O}^\times_{D,Q}$. Thus

$$\mathrm{Tr}_{D/C}(\tfrac{dz}{z}\alpha) = \mathrm{Tr}_{D/C}(\tfrac{dt}{t}\alpha) - \mathrm{Tr}_{D/C}(\tfrac{du}{u}\alpha) \equiv \tfrac{dt}{t}\mathrm{Tr}_{D/C}(\alpha) \mod \Omega^{q+1}_{C,P}.$$

Further by (S2) for Ω^q we have

$$s_P(\mathrm{Tr}_{D/C}(\alpha)) = e(Q/P)\mathrm{Tr}_{Q/P}(s_Q(\alpha)) = \mathrm{Tr}_{Q/P}(s_Q(\alpha)).$$

Hence we can take $\beta = \mathrm{Tr}_{D/C}(\alpha) \in \Omega^q_{C,P}$.

2. case: $e(Q/P) = p$, $f(Q/P) = 1$. In this case we have $t = z^p u$ for some $u \in \mathscr{O}^\times_{C,P} = \mathscr{O}^\times_{D,Q} \cap \kappa(C)$ (notice that $z^p \in \kappa(C)$). Further, since $\pi^* : \Omega^q_{C,P} \to \Omega^q_{D,Q}$ induces an isomorphism $\Omega^q_P \to \Omega^q_Q$, there exists a $\beta \in \Omega^q_{C,P}$ and $\gamma_i \in \Omega^{q-i}_{D,Q}$, $i = 0, 1$, such that

$$\alpha = \pi^*\beta + z\gamma_0 + dz \cdot \gamma_1,$$

in particular $s_Q(\alpha) = s_P(\beta)$. We obtain

$$\mathrm{Tr}_{D/C}(\tfrac{dz}{z}\alpha) \equiv \mathrm{Tr}_{D/C}(\tfrac{dz}{z})\beta \overset{(\mathrm{Tr7})}{\equiv} \tfrac{d(t/u)}{(t/u)}\beta \equiv \tfrac{dt}{t}\beta \mod \Omega^{q+1}_{C,P}.$$

Hence the lemma. □

2.5.7. **Example.** We have

$$\Omega^0 = \mathbb{G}_a.$$

3. First properties of the category of reciprocity functors

3.1. Lax Mackey functors with specialization map.

3.1.1. **Definition.** Let $\mathscr{M} : \mathrm{Reg}^{\leqslant 1}\mathrm{Cor}^{\mathrm{op}} \to R-\mathrm{mod}$ be a presheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$. A *subset* $\mathscr{R}$ *of* $\mathscr{M}$ is a collection of subsets $\mathscr{R}(X) \subset \mathscr{M}(X)$ for $X \in \mathrm{Reg}^{\leqslant 1}$. Given such a subset we define $< \mathscr{R} > (X)$ to be the R-submodule of $\mathscr{M}(X)$ generated by all $\mathscr{M}(\gamma)(a)$, with $\gamma \in \mathrm{Cor}(X, Y)$, $Y \in \mathrm{Reg}^{\leqslant 1}$, and $a \in \mathscr{R}(Y)$. Clearly $\mathrm{Reg}^{\leqslant 1} \ni X \mapsto < \mathscr{R} > (X)$ is a subpresheaf with transfers of $\mathscr{M}$. We define the *quotient* $\mathscr{M}/\mathscr{R}$ *of* $\mathscr{M}$ *by the subset* $\mathscr{R}$ to be the presheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$ given by

$$X \mapsto \frac{\mathscr{M}(X)}{< \mathscr{R} > (X)} =: \mathscr{M}/\mathscr{R}(X).$$

It is immediate that $\mathscr{M}/\mathscr{R}$ satisfies the following universal property: Any morphism $\Phi : \mathscr{M} \to \mathscr{N}$ of presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$ with $\Phi(\mathscr{R}(X)) = 0$ for all $X \in \mathrm{Reg}^{\leqslant 1}$ factors uniquely over $\mathscr{M}/\mathscr{R}$. In particular, if $\mathscr{R} \subset \mathscr{M}$ already is a subpresheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$, then $\mathscr{M}/\mathscr{R}$ is the usual quotient in the Abelian category of presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$.

We will need the following auxiliary category.

3.1.2. **Definition.** A *lax Mackey functor with specialization map* $\mathscr{L}$ is a presheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$ which satisfies the following condition:

(W.F.P.) For all $C \in (\mathscr{C}/S)$ with generic point η the natural map

$$\varinjlim_{U \ni \eta} \mathscr{L}(U) \twoheadrightarrow \mathscr{L}(\eta)$$

is surjective, where the limit is over all integral $U \in \mathrm{Reg}^{\leqslant 1}$ of dimension 1 and with generic point η.

We denote by **LMFsp** the full subcategory of **PT** whose objects are lax Mackey functors with specialization map.

Notice that the forgetful functor $\mathbf{MFsp} \to \mathbf{LMFsp}$ is fully faithful.

3.1.3. *Remark.* Note that the quotient of a lax Mackey functor with specialization map $\mathscr{L}$ by a subset, as defined in Definition 3.1.1, is again in the category **LMFsp**.

3.1.4. **Proposition.** *There exists a functor*

$$\Sigma : \mathbf{LMFsp} \to \mathbf{MFsp}.$$

such that the composition

$$\mathbf{MFsp} \xrightarrow{\text{forget}} \mathbf{LMFsp} \xrightarrow{\Sigma} \mathbf{MFsp}$$

is the identity functor and for any $\mathscr{L} \in \mathbf{LMFsp}$ *we have a functorial map*

$$\mathscr{L} \to \Sigma(\mathscr{L}) \quad \textit{in } \mathbf{LMFsp},$$

which Σ *maps to the identity on* $\Sigma(\mathscr{L})$ *in* **MFsp** *and which is surjective on Nisnevich stalks. In particular* Σ *is left adjoint to the natural functor* $\mathbf{MFsp} \to \mathbf{LMFsp}$*, with adjunction maps given by* $\mathscr{L} \to \Sigma(\mathscr{L})$*, for* $\mathscr{L} \in \mathbf{LMFsp}$ *and* $\mathscr{M} \xrightarrow{\mathrm{id}} \Sigma(\mathscr{M})$*, for* $\mathscr{M} \in \mathbf{MFsp}$.

Proof. Let $\mathscr{L}$ be a lax Mackey functor with specialization map. For $X \in \mathrm{Reg}^{\leqslant 1}$ with generic points η_i set

$$\mathscr{R}(X) := \mathrm{Ker}(\mathscr{L}(X) \to \oplus_i \mathscr{L}(\eta_i)).$$

Then $\mathscr{R}$ is a subset of $\mathscr{L}$ and we set

$$\mathscr{L}' := \mathscr{L}/\mathscr{R},$$

in the sense of Definition 3.1.1. Clearly we obtain in this way an endofunctor $\mathbf{LMFsp} \to \mathbf{LMFsp}$, $\mathscr{L} \mapsto \mathscr{L}'$, which is the identity on the full subcategory $\mathbf{MFsp}$ and the natural surjection of presheaves $\mathscr{L} \twoheadrightarrow \mathscr{L}'$ is functorial. Now for $\mathscr{L} \in \mathbf{LMFsp}$ we define recursively

$$\mathscr{L}^1 := \mathscr{L}', \quad \mathscr{L}^n := (\mathscr{L}^{n-1})' \quad , n \geqslant 2,$$

and set

$$\mathscr{L}^\infty = \varinjlim_n \mathscr{L}^n.$$

We obtain a functor $\mathbf{LMFsp} \to \mathbf{LMFsp}$, $\mathscr{L} \mapsto \mathscr{L}^\infty$, which is the identity on $\mathbf{MFsp}$ together with a functorial surjection $\mathscr{L} \twoheadrightarrow \mathscr{L}^\infty$. Further the restriction map $\mathscr{L}^\infty(U) \to \mathscr{L}^\infty(\eta)$ is injective for all integral $U \in \mathrm{Reg}^{\leqslant 1}$ with generic point η. Indeed, if $a \in \mathscr{L}^\infty(U)$ maps to zero in $\mathscr{L}^\infty(\eta)$, there exists a representative $a_n \in \mathscr{L}^n(U)$ of a such that a_n maps to zero in $\mathscr{L}^n(\eta)$. But then a_n maps to zero in $\mathscr{L}^{n+1}(U) = (\mathscr{L}^n)'(U)$ by definition and hence $a = 0 \in \mathscr{L}^\infty(U)$. Thus $\mathscr{L}^\infty$ satisfies the conditions (Inj) and (F.P.) from Definition 1.3.5. Hence

$$\Sigma(\mathscr{L}) := \mathscr{L}^\infty_{\mathrm{Nis}}$$

is a Mackey functor with specialization map by Lemma 1.3.7 and we obtain a functor Σ as in the statement. □

3.2. RF is a quasi-Abelian category.

3.2.1. **Proposition.** *The category* **MF** *is Abelian. If $\Phi : M \to N$ is a morphism of Mackey functors, then for all S-points x we have in R-mod*

$$(\mathrm{Ker}\,\Phi)(x) = \mathrm{Ker}(\Phi : M(x) \to N(x)), \quad (\mathrm{Coker}\,\Phi)(x) = N(x)/\Phi(M(x)),$$
$$(\mathrm{Im}\,\Phi)(x) = \Phi(M(x)), \quad (\mathrm{Coim}\,\Phi)(x) = M(x)/\mathrm{Ker}\,\Phi(x).$$

Proof. Straightforward. □

3.2.2. *Quasi-Abelian categories.* We recall the definition and some basic notions of quasi-Abelian categories from [29, 1.1]. Let $\mathscr{B}$ be an additive category that has kernels and cokernels. The image $\mathrm{Im}\, f$ of a map $f : X \to Y$ in $\mathscr{B}$ is then defined to be the kernel of $Y \to \mathrm{Coker}\, f$ and the coimage $\mathrm{Coim}\, f$ to be the cokernel of $\mathrm{Ker}\, f \to X$. The map f is called *strict* if the canonical map $\mathrm{Coim}\, f \to \mathrm{Im}\, f$ induced by f is an isomorphism. Following [29, Def. 1.1.3] we say that $\mathscr{B}$ is a *quasi-Abelian category* if the following two axioms are satisfied

- (QA) Let $f : X \to Y$ be a strict epimorphism, then the pullback $f' : X' \to Y'$ of f along any map $Y' \to Y$ is also a strict epimorphism.
- (QA*) Let $f : X \to Y$ be a strict monomorphism, then the pushout $f' : X' \to Y'$ of f along any map $X \to X'$ is also a strict monomorphism.

Now let $\mathscr{B}$ be a quasi-Abelian category. Then one says that a complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{B}$ is *strictly exact* (resp. *strictly coexact*) if f (resp. g) is strict and the natural map $\mathrm{Im}\, f \to \mathrm{Ker}\, g$ is an isomorphism. A complex $\ldots \to X_1 \to \ldots X_n \to \ldots$ is strictly exact (resp. coexact) if it is so at all spots $X_{i-1} \to X_i \to X_{i+1}$. It follows that a complex $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is strictly exact iff it is strictly coexact iff

$X \cong \operatorname{Ker} g$ (via f) and $\operatorname{Coker} f \cong Z$ (via g); we call complexes of this kind simply *short exact sequences*.

Finally, by [29, Def. 1.1.17 and 1.1.18], an additive functor $F : \mathscr{B} \to \mathscr{C}$ between quasi-Abelian categories is said to be *right exact* if it transforms any strictly coexact sequence $0 \to X \to Y \to Z \to 0$ into a strictly coexact sequence $F(X) \to F(Y) \to F(Z) \to 0$ (i.e. it preserves cokernels of strict morphisms) and *exact* if it preserves short exact sequences.

Next we want to show that $\mathbf{RF}$ is a quasi-Abelian category. For this we first clarify what kernels, cokernels, fibered products and - sums in $\mathbf{RF}$ are.

3.2.3. *Kernels.* Let $\Phi : \mathscr{M} \to \mathscr{N}$ be a morphism between reciprocity functors. Then it is straightforward to check that the kernel $\operatorname{Ker} \Phi$ in the Abelian category of presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$ is a reciprocity functor and hence is the kernel of Φ in the category $\mathbf{RF}$.

3.2.4. *Cokernels.* Let $\Phi : \mathscr{M} \to \mathscr{N}$ be a morphism between reciprocity functors. Denote by $\operatorname{Coker} \Phi$ the cokernel of Φ in $\mathbf{PT}$. Clearly Coker satisfies the condition (W.F.P.) from Definition 3.1.2 and hence is an object in $\mathbf{LMFsp}$. We set

$$\operatorname{Coker}_{\mathbf{RF}} \Phi := \Sigma(\operatorname{Coker} \Phi),$$

where $\Sigma : \mathbf{LMFsp} \to \mathbf{MFsp}$ is the functor from Proposition 3.1.4. Then it is straightforward to check that $\operatorname{Coker}_{\mathbf{RF}} \Phi$ is a reciprocity functor (use that $\mathscr{N} \to \operatorname{Coker}_{\mathbf{RF}} \Phi$ is surjective on Nisnevich stalks and Remark 1.5.2) and that the composition $\mathscr{N} \to \operatorname{Coker} \Phi \to \operatorname{Coker}_{\mathbf{RF}} \Phi$ is the cokernel of Φ in the category $\mathbf{RF}$ (use the left adjoint property of Σ).

Notice that the Mackey functor underlying $\operatorname{Coker}_{\mathbf{RF}} \Phi$ is in general only a quotient of the cokernel of the map on Mackey functors underlying Φ. The following Lemma gives a criterion, when they are the same.

3.2.5. **Lemma.** *Let $\Phi : \mathscr{M} \to \mathscr{N}$ be a map of reciprocity functors. For $X \in \mathrm{Reg}^{\leqslant 1}$ with generic points η_i define*

$$\mathscr{C}k(X) := \frac{\mathscr{N}(X)}{\mathscr{N}(X) \cap \oplus_i \Phi(\mathscr{M}(\eta_i))},$$

where the intersection in the denominator is taken inside $\oplus_i \mathscr{N}(\eta_i)$. Then $X \mapsto \mathscr{C}k(X)$ has a unique structure of presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$ such that the natural surjection $\mathscr{N} \twoheadrightarrow \mathscr{C}k$ is a map in $\mathbf{PT}$ if and only if the following condition is satisfied: For all $C \in (\mathscr{C}/S)$ and $P \in C$ we have (with the notation from 1.3.2)

$$s_P^{\mathscr{N}}(\mathscr{N}_{C,P} \cap \Phi(\mathscr{M}(\eta_C))) \subset \Phi(\mathscr{M}(P)). \tag{3.1}$$

Furthermore in this case $\mathscr{C}k$ satisfies (Inj) and (F.P.) and its Nisnevich sheafification $\mathscr{C}k_{\mathrm{Nis}}$ equals $\operatorname{Coker}_{\mathbf{RF}} \Phi$.

Proof. First of all notice that if $\mathscr{N} \to \mathscr{C}k$ is a map in $\mathbf{PT}$, then clearly $\mathscr{C}k$ satisfies (Inj) and (F.P.) and it is straightforward to check that $\mathscr{C}k_{\mathrm{Nis}}$ satisfies the universal property of the cokernel. Furthermore in this case the specialization maps $s_P^{\mathscr{N}} : \mathscr{N}_{C,P} \to \mathscr{N}(P)$ induce specialization maps $\mathscr{C}k_{C,P} \to \mathscr{C}k(P)$ and hence condition (3.1) has to be fulfilled. It remains to show that $\mathscr{N} \to \mathscr{C}k$ is a map in $\mathbf{PT}$ if (3.1) is satisfied. For this it suffices to show that for any elementary correspondence $[V] \in \operatorname{Cor}(X, Y)$ between integral schemes $X, Y \in \mathrm{Reg}^{\leqslant 1}$ the composition

$$\mathscr{N}(Y) \to \mathscr{N}(X) \to \mathscr{C}k(X)$$

factors over $\mathscr{C}k(Y)$. Further by Lemma 1.1.4, 1. it suffices to consider separately the two cases in which either the image of V in Y contains the generic point η_Y of Y or V is the graph of the inclusion of a closed point in an open subset of a curve $X =: P \hookrightarrow Y \subset C$ and hence $\mathscr{N}([V]) = s_P^{\mathscr{N}} : \mathscr{N}(Y) \to \mathscr{N}(P)$. But

in the first case $[V]$ restricts to a correspondence in $\mathrm{Cor}(\eta_X, \eta_Y)$ and hence maps $\mathscr{N}(Y) \cap \Phi(\mathscr{M}(\eta_Y))$ to $\mathscr{N}(X) \cap \Phi(\mathscr{M}(\eta_X))$, i.e. induces a map $\mathscr{C}k(Y) \to \mathscr{C}k(X)$; and in the second case this is simply the condition (3.1).

$\square$

3.2.6. *Fibered products.* Let $\mathscr{M} \xrightarrow{\Phi} \mathscr{L} \xleftarrow{\Psi} \mathscr{N}$ be two maps of reciprocity functors. Then it is straightforward to check that the fiber product $\mathscr{M} \times_{\mathscr{L}} \mathscr{N}$ in **PT** is a reciprocity functor and hence the fiber product in the category **RF**.

3.2.7. *Fibered sums.* Let $\mathscr{M} \xleftarrow{\Phi} \mathscr{P} \xrightarrow{\Psi} \mathscr{N}$ be two maps of reciprocity functors. Then

$$\mathrm{Reg}^{\leqslant 1} \ni X \mapsto (\mathscr{M} \oplus_{\mathscr{P}} \mathscr{N})(X) := \frac{\mathscr{M}(X) \oplus \mathscr{N}(X)}{\{(\Phi(a), -\Psi(a)) \mid a \in \mathscr{P}(X)\}}$$

is the fibered sum with respect to Φ and Ψ in **PT** and clearly lies in **LMFsp**. We set

$$\mathscr{M} \oplus^{\mathbf{RF}}_{\mathscr{P}} \mathscr{N} := \Sigma(\mathscr{M} \oplus_{\mathscr{P}} \mathscr{N}),$$

where Σ is the functor from Proposition 3.1.4. It is straightforward to check that $\mathscr{M} \oplus^{\mathbf{RF}}_{\mathscr{P}} \mathscr{N}$ is a reciprocity functor (use that the natural map $\mathscr{M} \oplus_{\mathscr{P}} \mathscr{N} \twoheadrightarrow \mathscr{M} \oplus^{\mathbf{RF}}_{\mathscr{P}} \mathscr{N}$ is surjective on Nisnevich stalks and Remark 1.5.2) and that it is in fact the fibered sum with respect to Φ and Ψ in the category **RF** (use the left adjoint property of Σ).

3.2.8. **Lemma.** *Let $\Phi : \mathscr{M} \to \mathscr{N}$ be a map of reciprocity functors. Then*

(1) Φ is a strict epimorphism in **RF** *iff it is a surjection of Nisnevich sheaves on $\mathrm{Reg}^{\leqslant 1}$ (i.e. surjective on Nisnevich stalks).*

(2) Φ is a strict monomorphism iff it is an injection of Nisnevich sheaves (i.e. an injection of presheaves) and for all $C \in (\mathscr{C}/S)$ and all $P \in C$ we have

$$\mathscr{N}_{C,P} \cap \Phi(\mathscr{M}(\eta_C)) \subset \Phi(\mathscr{M})^h_{C,P}, \tag{3.2}$$

i.e. for all open neighborhoods $U \subset C$ of P and all $a \in \mathscr{N}(U) \cap \Phi(\mathscr{M}(\eta_C))$ there exists a Nisnevich neighborhood $\pi : V \to U$ of P and an element $b \in \mathscr{M}(V)$ such that $\pi^*(a) = \Phi(b)$.

Proof. We start with a general remark: If $\Phi : \mathscr{M} \to \mathscr{N}$ is any map in **RF**, then clearly for all $P \in C$

$$s_P^{\mathscr{M}}(\mathscr{M}_{C,P} \cap \operatorname{Ker} \Phi(\eta_C)) \subset \operatorname{Ker} \Phi(P),$$

i.e. condition (3.1) holds for the map $\operatorname{Ker} \Phi \to \mathscr{M}$. Hence by Lemma 3.2.5 the coimage $\mathrm{Coim}_{\mathbf{RF}} \Phi = \mathrm{Coker}_{\mathbf{RF}}(\operatorname{Ker} \Phi \to M)$ is the Nisnevich sheafification of

$$X \mapsto \frac{\mathscr{M}(X)}{\mathscr{M}(X) \cap \oplus_i \operatorname{Ker} \Phi(\eta_i)}, \quad \text{with } \eta_i \text{ the generic points of } X.$$

(1). Assume Φ is a strict epimorphism in **RF**. Then $\mathrm{Coker}_{\mathbf{RF}} \Phi = 0$. Thus $\mathrm{Image}_{\mathbf{RF}} \Phi := \operatorname{Ker}(\mathscr{N} \to \mathrm{Coker}_{\mathbf{RF}} \Phi) = \mathscr{N}$ and Φ induces an isomorphism

$$\mathrm{Coim}_{\mathbf{RF}} \Phi \cong \mathscr{N}.$$

Thus the surjectivity of Φ as a map of Nisnevich sheaves follows from the description of the coimage above.

Now assume that Φ is a surjection as a morphism of Nisnevich sheaves. Then $(\mathrm{Coker}\ \Phi)_{\mathrm{Nis}} = 0$ a fortiori $\mathrm{Coker}_{\mathbf{RF}} \Phi = 0$, i.e. Φ is an epi in **RF** and $\mathrm{Image}_{\mathbf{RF}} \Phi = \mathscr{N}$. Further it follows easily from the description of the coimage above that the natural map $\mathrm{Coim}_{\mathbf{RF}} \Phi \to \mathscr{N}$ is an isomorphism, i.e. Φ is strict.

(2). Assume Φ is a strict monomorphism. Then $\operatorname{Ker} \Phi = 0$ and Φ induces an isomorphism

$$\mathscr{M} = \mathrm{Coim}_{\mathbf{RF}} \Phi \cong \mathrm{Image}_{\mathbf{RF}} \Phi. \tag{3.3}$$

Further for all integral $X \in \mathrm{Reg}^{\leqslant 1}$ with generic point η the R-submodule $\mathscr{N}(X) \cap (\Phi(\mathscr{M})(\eta))$ of $\mathscr{N}(X)$ is sent to zero in $\mathrm{Coker}_{\mathbf{RF}}\,\Phi$ (since the restriction map $\mathrm{Coker}_{\mathbf{RF}}\,\Phi(X) \to \mathrm{Coker}_{\mathbf{RF}}\,\Phi(\eta)$ is injective). Thus for all $P \in C$

$$\mathscr{N}_{C,P} \cap (\Phi(\mathscr{M})(\eta_C)) \subset \mathrm{Image}_{\mathbf{RF}}\Phi_{C,P} \subset \mathrm{Image}_{\mathbf{RF}}\Phi^h_{C,P}.$$

Moreover we have natural maps $\mathscr{N} \to (\mathrm{Coker}\ \Phi)_{\mathrm{Nis}} \to \mathrm{Coker}_{\mathbf{RF}}\,\Phi$ of Nisnevich sheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$, thus an inclusion in $\mathbf{NT}$

$$(\mathrm{Image}\ \Phi)_{\mathrm{Nis}} \hookrightarrow \mathrm{Image}_{\mathbf{RF}}\Phi, \tag{3.4}$$

where on the left hand side we consider the Nisnevich sheafification of the image of Φ in $\mathbf{PT}$ (it is in $\mathbf{NT}$ by Lemma 1.3.7). Since the precomposition of (3.4) with the map $\mathscr{M} \to (\mathrm{Image}\ \Phi)_{\mathrm{Nis}}$ is an isomorphism by (3.3), we obtain

$$(\mathrm{Image}\ \Phi)_{\mathrm{Nis}} \cong \mathrm{Image}_{\mathbf{RF}}\Phi.$$

Thus for all $P \in C$

$$\Phi(\mathscr{M})^h_{C,P} = (\mathrm{Image}\ \Phi)^h_{\mathrm{Nis},C,P} = (\mathrm{Image}^{\mathbf{RF}}\Phi)^h_{C,P} \supset (\mathscr{N}_{C,P} \cap \Phi(\mathscr{M}(\eta_C))).$$

Hence condition (3.2) is satisfied.

Now assume Φ is an injection of presheaves and condition (3.2) holds. Then $\mathrm{Ker}\ \Phi = 0$, i.e. Φ is a monomorphism in $\mathbf{RF}$. Furthermore condition (3.2) implies that Φ satisfies condition (3.1) and hence Lemma 3.2.5 gives a description of $\mathrm{Coker}_{\mathbf{RF}}\,\Phi$. It follows that $\mathrm{Image}_{\mathbf{RF}}\,\Phi$ is the Nisnevich sheafification of

$$X \mapsto \mathscr{N}(X) \cap \oplus_i \Phi(\mathscr{M}(\eta_i)) \quad \text{with } \eta_i \text{ the generic points of } X.$$

Hence condition (3.2) together with the injectivity of Φ imply that Φ induces an isomorphism

$$\mathscr{M} = \mathrm{Coim}_{\mathbf{RF}}\Phi \xrightarrow{\simeq} \mathrm{Image}_{\mathbf{RF}}\Phi,$$

i.e. Φ is strict. □

3.2.9. **Theorem.** *The category* $\mathbf{RF}$ *of reciprocity functors is quasi-Abelian (see 3.2.2). Furthermore the forgetful (or restriction) functor* $\mathbf{RF} \to \mathbf{MF}$ *is exact (in the sense of 3.2.2).*

Proof. We have to check that the two conditions (QA) and (QA*) from 3.2.2 are satisfied. For (QA) this follows easily from the explicit description of strict epimorphisms in Lemma 3.2.8 above and the explicit description of fibered products in 3.2.6. For (QA*) assume we are given a pair of maps $\mathscr{M} \xleftarrow{\Psi} \mathscr{P} \xrightarrow{\Phi} \mathscr{N}$ in $\mathbf{RF}$, with Φ a strict monomorphism. Then it follows from the explicit description of strict monomorphisms in the lemma above that the map

$$\Pi : \mathscr{P} \to \mathscr{M} \oplus \mathscr{N}, \quad a \mapsto (\Phi(a), -\Psi(a))$$

satisfies condition (3.1). (Indeed, assume we are given $a \in \mathscr{M}_{C,P}$, $b \in \mathscr{N}_{C,P}$ and $c \in \mathscr{P}(\eta_C)$ with $a = \Phi(c)$ and $b = -\Psi(c)$, then by (3.2) there exists a $c' \in \mathscr{P}^h_{C,P}$ such that $a = \Phi(c) = \Phi(c')$ in $\mathscr{M}^h_{C,P}$ and since Φ is injective we have $c = c' \in \mathscr{P}^h_{C,P}$ and thus $b = -\Psi(c')$ in $\mathscr{N}^h_{C,P}$; hence $s_P(a,b) = \Pi(s_P(c'))$.) Therefore by Lemma 3.2.5 $\mathscr{M} \oplus^{\mathbf{RF}}_{\mathscr{P}} \mathscr{N} = \mathrm{Coker}_{\mathbf{RF}}\,\Pi$ is the Nisnevich sheaf associated to

$$X \mapsto \frac{\mathscr{M}(X) \oplus \mathscr{N}(X)}{(\mathscr{M}(X) \oplus \mathscr{N}(X)) \cap \oplus_i \Pi(\mathscr{P}(\eta_i))}.$$

We have to check that (QA*) holds i.e. that $\Phi' : \mathscr{N} \to \mathscr{M} \oplus^{\mathbf{RF}}_{\mathscr{P}} \mathscr{N}$ is a strict monomorphism. Using the above description and the injectivity of Φ we see that Φ' is injective. It remains to check condition (3.2) for Φ'. So take $P \in C$, an integral Nisnevich neighborhood U of P and assume we are given elements $a \in \mathscr{M}(U)$,

$b \in \mathscr{N}(U)$ and $c \in \mathscr{N}(\eta_C)$ such that $(a,b) = (0,c)$ in $(\mathscr{M} \oplus^{\mathbf{RF}}_{\mathscr{P}} \mathscr{N})(\eta_U)$. Then there exists an element $d \in \mathscr{P}(\eta_U)$ such that

$$(a, b-c) = (\Phi(d), -\Psi(d)) \quad \text{in } \mathscr{M}(\eta_U) \oplus \mathscr{N}(\eta_U).$$

But since a already lies in $\mathscr{M}(U)$ it follows from condition (3.2) for Φ, that there exists a Nisnevich neighborhood $V \to U$ of P and an element $e \in \mathscr{P}(V)$ such that $a = \Phi(d) = \Phi(e)$ in $\mathscr{M}(V)$. Since Φ is injective we have $d = e \in \mathscr{P}(V)$ and thus $b - c = -\Psi(e) \in \mathscr{N}(V)$, i.e. $\Phi'(c) = (0,c) \in \Phi'(\mathscr{N}(V))$, i.e. condition (3.2) is satisfied. Thus **RF** is quasi-Abelian.

For the last statement let $0 \to \mathscr{L} \xrightarrow{\Phi} \mathscr{M} \xrightarrow{\Psi} \mathscr{N} \to 0$ be a short exact sequence in **RF**, i.e. $\mathscr{L} \cong \operatorname{Ker}\Psi$ and $\operatorname{Coker}_{\mathbf{RF}}\Phi \cong \mathscr{N}$. Then clearly also $\mathscr{L} \cong \operatorname{Ker}\Psi$ in **MF**. Further to show $\operatorname{Coker}_{\mathbf{RF}}\Phi \cong \mathscr{N}$ in **MF** it suffices by Lemma 3.2.5 to check condition (3.1). So take $C \in (\mathscr{C}/S)$ and $P \in C$ and $a \in \mathscr{M}_{C,P} \cap \Phi(\mathscr{L}(\eta_C))$. Then $\Psi(a) = 0$ and hence $a \in \mathscr{M}_{C,P} \cap \operatorname{Ker}\Psi \cong \Phi(\mathscr{L}_{C,P})$, thus $s_P^{\mathscr{M}}(a) \in \Phi(\mathscr{L}(P))$. This finishes the proof. □

3.2.10. *Remark.* The above proof together with Lemma 3.2.5 in particular implies that if $0 \to \mathscr{L} \xrightarrow{\Phi} \mathscr{M} \xrightarrow{\Psi} \mathscr{N} \to 0$ is a short exact sequence in **RF**, then $\mathscr{N} \cong \operatorname{Coker}_{\mathbf{RF}}\Phi$ is the Nisnevich sheafification of

$$X \mapsto \frac{\mathscr{M}(X)}{\mathscr{M}(X) \cap \oplus_i \Phi(\mathscr{L}(\eta_i))}, \quad \text{with } \eta_i \text{ the generic points of } X.$$

3.2.11. *Remark.* All statements and proofs from 3.2.3 to 3.2.10 above work verbatim with **RF** replaced by **MFsp**.

3.2.12. **Lemma.** *Let*

$$0 \to H \xrightarrow{a} G \xrightarrow{b} Q \to 0$$

be a short exact sequence of smooth, commutative, connected group schemes over S. Assume that for all integral $X \in \operatorname{Reg}^{\leqslant 1}$ and all $x \in X$ (closed or not) the map a induces an injection

$$H^1_{\mathrm{fppf}}(\operatorname{Spec}\mathscr{O}^h_{X,x}, H) \hookrightarrow H^1_{\mathrm{fppf}}(\operatorname{Spec}\mathscr{O}^h_{X,x}, G). \tag{3.5}$$

Then the corresponding sequence of reciprocity functors (see Proposition 2.2.2) is a short exact sequence in the sense of 3.2.2.

Proof. Clearly $\operatorname{Ker} b \cong H$ in **RF**. By our injectivity assumption (3.5) we have

$$Q(x) \cong G(x)/H(x) \quad \text{and} \quad Q(\mathscr{O}^h_{C,P}) \cong G(\mathscr{O}^h_{C,P})/H(\mathscr{O}^h_{C,P})$$

for all S-points x and all $C \in (\mathscr{C}/S)$, $P \in C$. Hence the fact that the natural maps $Q(U) \hookrightarrow Q(\eta_U)$, $U \in \operatorname{RegCon}^{\leqslant 1}$, are inclusions yields an isomorphism for all $C \in (\mathscr{C}/S)$ and all $P \in C$

$$\varinjlim_{U \ni P} (G(U) \cap H(\eta_U)) \cong H(\mathscr{O}^h_{C,P}),$$

where the limit is over all integral Nisnevich neighborhoods of P. In particular the condition (3.1) is satisfied. Thus the cokernel formula in Lemma 3.2.5 says that the natural map $\operatorname{Coker}_{\mathbf{RF}} a \to Q$ is an isomorphism on Nisnevich stalks and hence an isomorphism in **RF**. This finishes the proof. □

3.3. **Truncated reciprocity functors.**

3.3.1. **Lemma.** *For all $n \geqslant 1$, the forgetful functor $o_n : \mathbf{RF}_n \to \mathbf{LMFsp}$ has a left adjoint*

$$\varrho_n : \mathbf{LMFsp} \to \mathbf{RF}_n$$

such that $\varrho_n \circ o_n = \mathrm{id}$.

Proof. Let $\mathscr{M} \in \mathbf{LMFsp}$ be a lax Mackey functor with specialization map. Consider the quotient as presheaves with transfers (in the sense of 3.1.1)

$$\mathrm{L}\varrho_n(\mathscr{M}) := \mathscr{M}/\mathcal{R}_n$$

of $\mathscr{M}$ by the subset $\mathcal{R}_n$ consisting of the elements

$$\sum_{P \in U} v_P(f) \cdot \mathrm{Tr}_{P/x_C}(s_P^{\mathscr{M}}(a)),$$

where $C \in (\mathscr{C}/S)$ is a curve, $U \subseteq C$ is a non-empty open subset, $a \in \mathscr{M}(U)$ and $f \in \mathbb{G}_m(\eta_C)$ with $f \equiv 1$ modulo $\sum_{P \in C\setminus U} n[P]$. Then $\mathrm{L}\varrho_n(\mathscr{M})$ is a lax Mackey functor with specialization, and we define a Mackey functor with specialization map

$$\varrho_n(\mathscr{M}) := \Sigma(\mathrm{L}\varrho_n(\mathscr{M})),$$

with Σ as in Proposition 3.1.4. It follows from Theorem 1.4.8 and Lemma 1.5.10 that $\varrho_n(\mathscr{M})$ is a reciprocity functor which lies in $\mathbf{RF}_n$. Clearly, if $\mathscr{M} \in \mathbf{RF}_n$, then $\mathcal{R}_n = 0$ and thus $\varrho_n \circ o_n(\mathscr{M}) = \Sigma(o_n(\mathscr{M})) = \mathscr{M}$, by Proposition 3.1.4. Finally by the left adjoint property of Σ we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{RF}_n}(\varrho_n(\mathscr{M}), \mathscr{N}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{LMFsp}}(\mathrm{L}\varrho_n(\mathscr{M}), o_n(\mathscr{N})) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{LMFsp}}(\mathscr{M}, o_n(\mathscr{N})).\end{aligned}$$

Hence ϱ_n is left adjoint to o_n. □

3.3.2. **Example.** Here is an example to show that the truncation functor can be quite brutal: $\varrho_1(\mathbb{G}_a) = 0$. Indeed, if $x = \mathrm{Spec}\, k$ is an S-point and we take $a \in k\setminus\{0\}$, we consider $\mathbb{A}^1 = \mathrm{Spec}\, k[t] \subset \mathbb{P}^1$ and the function $f = t/(t+a) \in k(t)^\times$, which is congruent to 1 modulo $\{\infty\}$. Then the image of

$$\sum_{P \in \mathbb{A}^1} v_P(f) \mathrm{Tr}_{P/x} s_P(t+a) = a$$

is zero in $\varrho_1(\mathbb{G}_a)$.

In particular the left adjoint property of ϱ_1 implies that any map of reciprocity functors $\mathbb{G}_a \to \mathscr{M}$ with $\mathscr{M} \in \mathbf{RF}_1$ is the zero map.

4. K-groups of reciprocity functors

4.1. **Tensor products in PT.**

4.1.1. For $X \in \mathrm{RegCon}^{\leqslant 1}$, we set

$$(\mathscr{M} \otimes \mathscr{N})(X) := \left(\bigoplus_{Y \xrightarrow{\text{fin. fl.}} X} \mathscr{M}(Y) \otimes_R \mathscr{N}(Y) \right) / \mathcal{R}(X)$$

where the sum is taken over all finite flat morphisms $Y \to X$ in $\mathrm{RegCon}^{\leqslant 1}$ and $\mathcal{R}(X)$ is the submodule of the direct sum generated by the elements

$$(a \otimes g_* b') - (g^* a \otimes b'), \qquad (g_* a' \otimes b) - (a' \otimes g^* b)$$

where $g : Y' \to Y$ is a finite flat morphism over X and $a \in \mathscr{M}(Y)$, $a' \in \mathscr{M}(Y')$, $b \in \mathscr{N}(Y)$, $b' \in \mathscr{N}(Y')$. For $X \in \mathrm{Reg}^{\leqslant 1}$ we extend the definition additively.

4.1.2. **Lemma.** *Let $\mathscr{M}, \mathscr{N} \in \mathbf{PT}$ be two presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$. Then $\mathscr{M} \otimes \mathscr{N}$ is canonically a presheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$.*

Proof. It is enough to check that $\mathscr{M} \otimes \mathscr{N}$ is equipped with pullback maps and pushforward maps on $\mathrm{RegCon}^{\leqslant 1}$ satisfying the conditions of Lemma 1.2.2.

Pushforward: Let $f : X \to Y$ be a finite flat morphism in $\mathrm{RegCon}^{\leqslant 1}$. Then the natural inclusions, for finite flat morphisms $X' \to X$ in $\mathrm{RegCon}^{\leqslant 1}$,

$$\mathscr{M}(X') \otimes_R \mathscr{N}(X') \to \bigoplus_{Y' \xrightarrow{\text{fin. fl.}} Y} \mathscr{M}(Y') \otimes_R \mathscr{N}(Y')$$

induce a morphism of R-modules

$$f_* : (\mathscr{M} \otimes \mathscr{N})(X) \to (\mathscr{M} \otimes \mathscr{N})(Y),$$

which clearly is functorial.

Pullback: Let $g : Y \to X$ be a morphism in $\mathrm{RegCon}^{\leqslant 1}$. Given a finite flat morphism $X' \to X$ in $\mathrm{RegCon}^{\leqslant 1}$, consider the R-linear morphism

$$g^*_{X'} : \mathscr{M}(X') \otimes_R \mathscr{N}(X') \to (\mathscr{M} \otimes \mathscr{N})(Y), \quad a \otimes b \mapsto \sum_{T \subseteq Y'} \lg(\mathscr{O}_{Y',\eta_T}) \cdot (g_T^* a \otimes g_T^* b), \tag{4.1}$$

where the sum is taken over the irreducible components of $Y' = Y \times_X X'$ and $g_T : \tilde{T} \to X'$ is the canonical morphism with $\tilde{T}$ the normalization of T. Note that the definition makes sense since the canonical morphism $f_T : \tilde{T} \to Y$ is finite and flat. Let us show that these morphisms induce an R-linear morphism

$$g^* : (\mathscr{M} \otimes \mathscr{N})(X) \to (\mathscr{M} \otimes \mathscr{N})(Y).$$

Let $f : X'' \to X'$ be a finite flat X-morphism between two finite flat connected X-schemes in $\mathrm{Reg}^{\leqslant 1}$. Let us use the following notation

$$\begin{tikzcd}
\tilde{T}' \arrow[r, "f'"] \arrow[d, "\nu_{T'}"'] \arrow[ddd, bend right=60, "g_{T'}"'] & \tilde{T} \arrow[d, "\nu_T"] \arrow[ddd, dotted, bend left=60, "g_T"] & \\
T' \arrow[r] \arrow[d] & T \arrow[d] & \\
Y'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "g"] \\
X'' \arrow[r, "f"] & X' \arrow[r] & X
\end{tikzcd}$$

and

$$l_{T'} = \lg(\mathscr{O}_{Y'',\eta_{T'}}), \quad l_T = \lg(\mathscr{O}_{Y',\eta_T}), \quad l_T^{T'} = \lg(\mathscr{O}_{X'' \times_{X'} T,\eta_{T'}})$$

where T is an irreducible component of Y' and T' is an irreducible component of Y'' that dominates T. By [7, Lemma A.4.1], applied to the flat local homomorphism $\mathscr{O}_{Y',\eta_T} \to \mathscr{O}_{Y'',\eta_{T'}}$, we have

$$l_{T'} = l_T \cdot l_T^{T'}. \tag{4.2}$$

Note that since f is universally equidimensional (being finite and flat), any irreducible component T of Y' is dominated by an irreducible component T' of Y''.

Now for $a' \in \mathscr{M}(X'')$ and $b \in \mathscr{N}(X')$ we have in $(\mathscr{M} \otimes \mathscr{N})(Y)$

$$\begin{aligned} g^*_{X'}((f_*a') \otimes b) &= \sum_{T \subseteq Y'} l_T \cdot (g^*_T f_* a' \otimes g^*_T b), && \text{by definition} \\ &= \sum_{T \subseteq Y'} \sum_{T' \subseteq X'' \times_{X'} T} l_T \cdot l_T^{T'} \cdot (f'_* g^*_{T'} a' \otimes g^*_T b), && \text{by Lemma 1.1.4} \\ &= \sum_{T' \subseteq Y''} l_{T'} \cdot (f'_* g^*_{T'} a' \otimes g^*_T b), && \text{by (4.2)} \\ &= \sum_{T' \subseteq Y''} l_{T'} \cdot (g^*_{T'} a' \otimes g^*_{T'} f^* b), && \text{by definition of } \mathcal{R}(Y) \\ &= g^*_{X''}(a' \otimes f^* b), && \text{by definition.} \end{aligned}$$

By symmetry we obtain that $g^*_{X'}$ maps $\mathcal{R}(X)$ to zero. This shows that the map (4.1) is well defined. Functoriality is proven by a similar computation.

Degree formula and cartesian square formula: Consider a finite and flat morphism $g : Y \to X$ in $\mathrm{RegCon}^{\leqslant 1}$. Let $f : X' \to X$ be a finite flat morphism in $\mathrm{RegCon}^{\leqslant 1}$, and $a \in \mathscr{M}(X')$, $b \in \mathscr{N}(X')$. Since g_T is a finite flat morphism (with notation as above), the projection formula gives in $(\mathscr{M} \otimes \mathscr{N})(X)$

$$\begin{aligned} g_* g^*(a \otimes b) = g^*_{X'}(a \otimes b) &= \sum_{T \subseteq Y'} \lg(\mathscr{O}_{Y',\eta_T}) \cdot g^*_T a \otimes g^*_T b \\ &= \sum_{T \subseteq Y'} \lg(\mathscr{O}_{Y',\eta_T}) \cdot a \otimes g_{T*} g^*_T b = \left(\sum_{T \subseteq Y'} \lg(\mathscr{O}_{Y',\eta_T}) \deg(g_T) \right) \cdot a \otimes b \\ &= \deg(g) \cdot a \otimes b. \end{aligned}$$

It remains to check the cartesian square formula. Let $g : Y \to X$ be a morphism in $\mathrm{RegCon}^{\leqslant 1}$ and $f : X' \to X$ be a finite flat morphism in $\mathrm{RegCon}^{\leqslant 1}$ and take $a \in \mathscr{M}(X')$ and $b \in \mathscr{N}(X')$. Using the above notation and (4.2) we obtain

$$\begin{aligned} \sum_{T \subseteq Y'} l_T \cdot f_{T*} g^*_T (a \otimes b) &= \sum_{T \subseteq Y'} l_T \cdot \left(\sum_{T' \subseteq X'' \times_{X'} T} l_T^{T'} \, (g^*_{T'} a \otimes g^*_{T'} b) \right) \\ &= \sum_{T' \subseteq Y''} l_{T'} \cdot (g^*_{T'} a \otimes g^*_{T'} b) = g^* f_* (a \otimes b), \end{aligned}$$

as desired. □

4.1.3. *Remark.* Let M and N be two Mackey functors (see Definition 1.3.1). Recall that the tensor product as Mackey functor of M and N is defined as follows (see e.g. [15]). Let x be an S-point

$$(M \overset{\mathrm{M}}{\otimes} N)(x) := \left[\bigoplus_{\xi \xrightarrow{\text{fin.}} x} M(\xi) \otimes_R N(\xi) \right] / \mathcal{R}(x)$$

where the sum is taken over all finite morphisms of S-points and $\mathcal{R}(x)$ is the submodule of the direct sum generated by the elements of the shape

$$(\delta_* a') \otimes b - a' \otimes \delta^* b \quad \text{and} \quad a \otimes \delta_* b' - \delta^* a \otimes b',$$

where $a \in M(\xi)$, $a' \in M(\xi')$, $b \in N(\xi)$, $b' \in M(\xi')$ and $\delta : \xi' \to \xi$ is a morphism between finite x-points. It follows from the definition that the underlying Mackey functor of the tensor product in **PT** is simply the tensor product of the underlying Mackey functors as defined above.

4.1.4. **Lemma.** *If* $\mathscr{M} \in \mathbf{LMFsp}$ *and* $\mathscr{N} \in \mathbf{LMFsp}$ *then* $\mathscr{M} \otimes \mathscr{N}$ *is also a lax Mackey functor with specialization map.*

Proof. Let C be a curve in $(\mathscr{C}/S)$. We have to show that the condition (W.F.P.) holds *i.e.* that the map

$$\operatorname*{colim}_{U \ni \eta_C} (\mathscr{M} \otimes \mathscr{N})(U) \to (\mathscr{M} \otimes \mathscr{N})(\eta_C) \tag{4.3}$$

is surjective. Let $\xi \to \eta_C$ be a *finite* morphism, and $a \in \mathscr{M}(\xi)$, $b \in \mathscr{N}(\xi)$. There exists an open subset V in the normalization C_ξ of C in $\xi \to \eta_C$ such that a lifts to $\mathscr{M}(V)$ and b to $\mathscr{N}(V)$. We may then find an open subset U in C such that V contains the image inverse U_ξ of U in C_ξ. Since the morphism $U_\xi \to U$ is finite and flat, we see that $a \otimes b$ lifts to $(\mathscr{M} \otimes \mathscr{N})(U)$, and therefore (4.3) is surjective. □

4.1.5. **Definition.** Let $\mathscr{M}_i$, $i = 1, \dots, n$, and $\mathscr{N}$ be objects in the category $\mathbf{PT}$ *i.e.* presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$. Then an *$n$-linear map of presheaves with transfers*

$$\Phi : \prod_{i=1}^{n} \mathscr{M}_i \to \mathscr{N}$$

is a collection of n-linear maps of R-modules $\Phi_X : \prod_{i=1}^{n} \mathscr{M}_i(X) \to \mathscr{N}(X)$, where X runs through all schemes $X \in \mathrm{RegCon}^{\leqslant 1}$, satisfying the following properties (we simply write Φ instead of Φ_X):

(L1) For $f : X \to Y$ a morphism of schemes in $\mathrm{RegCon}^{\leqslant 1}$ and $a_i \in \mathscr{M}_i(Y)$, $i = 1, \dots, n$ we have

$$f^*\Phi(a_1, \dots, a_n) = \Phi(f^*a_1, \dots, f^*a_n).$$

(L2) For $f : X \to Y$ a finite flat morphism of schemes in $\mathrm{RegCon}^{\leqslant 1}$ and $a_i \in \mathscr{M}_i(Y)$, $i \neq i_0$, and $b \in \mathscr{M}_{i_0}(X)$ we have

$$\begin{aligned}\Phi(a_1, \dots, a_{i_0-1}, f_*b, a_{i_0+1}, \dots, a_n)& \\ = f_*\Phi(f^*a_1, \dots f^*a_{i_0-1}, b, f^*a_{i_0+1}, \dots, f^*a_n)&.\end{aligned}$$

We denote by $n-\mathrm{Lin}(\mathscr{M}_1, \dots, \mathscr{M}_n; \mathscr{N})$ the R-module of n-linear maps $\prod_i \mathscr{M}_i \to \mathscr{N}$ and set $\mathrm{Bil}(\mathscr{M}_1, \mathscr{M}_2; \mathscr{N}) := 2 - \mathrm{Lin}(\mathscr{M}_1, \mathscr{M}_2; \mathscr{N})$.

4.1.6. **Corollary.** *For* $\mathscr{M}, \mathscr{N} \in \mathbf{PT}$ *the map* $\mathscr{M}(X) \times \mathscr{N}(X) \to (\mathscr{M} \otimes \mathscr{N})(X)$, $(a, b) \mapsto a \otimes b$, $X \in \mathrm{RegCon}^{\leqslant 1}$, *is a bilinear map and it is universal, i.e. it induces an isomorphism of functors on* $\mathbf{PT}$

$$\mathrm{Hom}_{\mathbf{PT}}(\mathscr{M} \otimes \mathscr{N}, -) \simeq \mathrm{Bil}(\mathscr{M}, \mathscr{N}; -).$$

Furthermore, the functor $\otimes$ *makes* $\mathbf{PT}$ *a monoidal category with unit object the constant presheaf* R.

Proof. Straightforward. □

4.1.7. **Example.** Let $\mathscr{M}$ be a presheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$ and N an R-module viewed as a constant presheaf with transfers (cf. Example 2.1). Then it is easy to check that $\mathscr{M} \otimes N$ is the presheaf with transfers given by $X \mapsto \mathscr{M}(X) \otimes_R N$, $X \in \mathrm{RegCon}^{\leqslant 1}$, and a correspondence $\alpha \in \mathrm{Cor}(X, Y)$ between two schemes in $\mathrm{RegCon}^{\leqslant 1}$ acts via $\mathscr{M}(\alpha) \otimes \mathrm{id}_N$.

4.2. **K-groups of reciprocity functors.**

4.2.1. **Definition.** Let $\mathscr{M}_i$, $i = 1, \dots, n$, and $\mathscr{N}$ be reciprocity functors. Then an *n-linear map of reciprocity functors*

$$\Phi : \prod_{i=1}^{n} \mathscr{M}_i \to \mathscr{N}$$

is an n-linear map in **PT** (see Definition 4.1.5) satisfying the following additional property:

(L3) For any sequence of positive integers $r_1, \dots, r_n \geqslant 1$, any $C \in (\mathscr{C}/S)$ and $P \in C$ we have

$$\Phi(\mathrm{Fil}_P^{r_1}\mathscr{M}_1(\eta_C) \times \ldots \times \mathrm{Fil}_P^{r_n}\mathscr{M}_n(\eta_C)) \subset \mathrm{Fil}_P^{\max\{r_1,\dots,r_n\}}\mathscr{N}(\eta_C),$$

where Fil_P is the filtration defined in Definition 1.5.6.

We denote by $n-\mathrm{Lin}_{\mathbf{RF}}(\mathscr{M}_1, \dots, \mathscr{M}_n; \mathscr{N})$ the R-module of n-linear maps $\prod_i \mathscr{M}_i \to \mathscr{N}$ and set $\mathrm{Bil}_{\mathbf{RF}}(\mathscr{M}_1, \mathscr{M}_2; \mathscr{N}) := 2 - \mathrm{Lin}_{\mathbf{RF}}(\mathscr{M}_1, \mathscr{M}_2; \mathscr{N})$.

Notice that if $\Psi_i : \mathscr{M}_i' \to \mathscr{M}_i$, $i = 1, \dots, n$, and $\Psi : \mathscr{N} \to \mathscr{N}'$ are morphisms of reciprocity functors and Φ is an n-linear map as above, then $\Phi \circ (\prod_i \Psi_i)$ and $\Psi \circ \Phi$ are also an n-linear maps. Thus $n-\mathrm{Lin}_{\mathbf{RF}}(\mathscr{M}_1, \dots, \mathscr{M}_n; \mathscr{N})$ has the expected functorial properties.

4.2.2. *Remark.* It follows from Lemma 1.5.9 that condition (L3) is equivalent to the following: For all $C \in (\mathscr{C}/S)$ and effective divisors $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ on C we have

$$\Phi(\mathscr{M}(C, \mathfrak{m}_1) \times \ldots \times \mathscr{M}(C, \mathfrak{m}_n)) \subset \mathscr{N}(C, \max\{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}).$$

4.2.3. **Definition.** Let $\mathscr{M}_i$, $i = 1, \dots, n$ be reciprocity functors.

(1) Let

$$\mathrm{LT}(\mathscr{M}_1, \dots, \mathscr{M}_n) := \mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_n / \mathcal{R},$$

be the quotient (in the sense of 3.1.1) of the presheaf with transfers $\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_n \in \mathbf{PT}$ by the subset $\mathcal{R}$, consisting of the elements

$$\sum_{P \in C \setminus |\max_i\{\mathfrak{m}_i\}|} v_P(f) \cdot s_P^{\mathscr{M}_1}(a_1) \otimes \ldots \otimes s_P^{\mathscr{M}_n}(a_n) \quad \text{in } (\mathscr{M}_1 \otimes \ldots \otimes \mathscr{M}_n)(x_C), \tag{4.4}$$

where $C \in (\mathscr{C}/S)$, $\mathfrak{m}_i$, $i = 1, \dots, n$, are effective divisors on C, $a_i \in \mathscr{M}_i(C, \mathfrak{m}_i)$ and $f \in \mathbb{G}_m(\eta_C)$ with $f \equiv 1 \bmod \max_i\{\mathfrak{m}_i\}$.

(2) By Remark 3.1.3 and Lemma 4.1.4, $\mathrm{LT}(\mathscr{M}_1, \dots, \mathscr{M}_n)$ lies in **LMFsp**, and the reciprocity K-functor associated to $\mathscr{M}_1, \dots, \mathscr{M}_n$ is defined by

$$\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n) := \Sigma(\mathrm{LT}(\mathscr{M}_1, \dots, \mathscr{M}_n)),$$

where Σ is the functor from Proposition 3.1.4.

4.2.4. **Theorem.** *Let $\mathscr{M}_i$, $i = 1, \dots, n$, be reciprocity functors. Then the functor $\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n)$ defined above is a reciprocity functor and the natural n-linear morphism of presheaves with transfers (in the sense of Definition 4.1.5)*

$$\mathscr{M}_1 \times \cdots \times \mathscr{M}_n \xrightarrow{\otimes} \mathscr{M}_1 \otimes \ldots \otimes \mathscr{M}_n \twoheadrightarrow \mathrm{LT}(\mathscr{M}_1, \dots, \mathscr{M}_n) \to \mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n) \tag{4.5}$$

is an n-linear morphism of reciprocity functors denoted by

$$\tau : \mathscr{M}_1 \times \cdots \times \mathscr{M}_n \to \mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n), \quad (a_1, \dots, a_n) \mapsto \tau(a_1, \dots, a_n). \tag{4.6}$$

Furthermore, this gives a functor

$$\prod_{i=1}^{n} \mathbf{RF} \to \mathbf{RF}, \quad (\mathscr{M}_1, \dots, \mathscr{M}_n) \mapsto \mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n),$$

which represents the functor $\mathbf{RF} \to (R-\mathrm{mod})$*,* $\mathscr{N} \mapsto n - \mathrm{Lin}_{\mathbf{RF}}(\mathscr{M}_1, \ldots, \mathscr{M}_n; \mathscr{N})$ *with* τ *as the universal* n*-linear map; in particular*

$$n - \mathrm{Lin}_{\mathbf{RF}}(\mathscr{M}_1, \ldots, \mathscr{M}_n; \mathscr{N}) = \mathrm{Hom}_{\mathbf{RF}}(T(\mathscr{M}_1, \ldots, \mathscr{M}_n), \mathscr{N}).$$

Proof. First we prove that $\mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n)$ is a reciprocity functor, i.e. we have to show that for $C \in (\mathscr{C}/S)$ a curve and $U \subset C$ a non-empty open subset any section $\alpha \in \mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n)(U)$ admits a modulus whose support is equal to $C \setminus U$. First assume that $\alpha = a_1 \otimes \ldots \otimes a_n$ with $a_i \in \mathscr{M}_i(U_\xi)$, where $\xi \to \eta_C$ is a finite morphism and we denote by C_ξ the normalization of C in $\kappa(\xi)$ and by U_ξ the pullback of U to C_ξ. Since the $\mathscr{M}_i$'s are reciprocity functors we find effective divisors $\mathfrak{m}_i$ on C with $|\mathfrak{m}_i| = C \setminus U$, such that $a_i \in \mathscr{M}_i(C_\xi, \mathfrak{m}_{i,\xi})$, where $\mathfrak{m}_{i,\xi}$ denotes the pullback of $\mathfrak{m}_i$ to C_ξ. Then $|\max_i\{\mathfrak{m}_i\}| = C \setminus U$ and we have to show

$$\sum_{P \in C \setminus |\max_i\{\mathfrak{m}_i\}|} v_P(f) \mathrm{Tr}_{P/x_C} s_P(a_1 \otimes \ldots \otimes a_n) = 0 \quad \text{in } \mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n)(x_C),$$

for all $f \in \mathbb{G}_m(\eta_C)$ with $f \equiv 1 \bmod \max_i\{\mathfrak{m}_i\}$. But by definition of Tr_{P/x_C} and s_P (see the construction of the pushforward and pullback in Lemma 4.1.2), we have that the sum on the left-hand side equals

$$\begin{aligned}\sum_{P \in C \setminus |\max_i\{\mathfrak{m}_i\}|} v_P(f) \sum_{P' \in C_\xi \times_C P} & e(P'/P)(s_{P'}^{\mathscr{M}_1}(a_1) \otimes \ldots \otimes s_{P'}^{\mathscr{M}_n}(a_n)) \\ &= \sum_{P' \in C_\xi \setminus |\max_i\{\mathfrak{m}_{i,\xi}\}|} v_{P'}(f)(s_{P'}^{\mathscr{M}_1}(a_1) \otimes \ldots \otimes s_{P'}^{\mathscr{M}_n}(a_n)),\end{aligned}$$

which is zero in $\mathrm{LT}(\mathscr{M}_1, \ldots, \mathscr{M}_n)(x_C)$ by the relation (4.4) we divide out; a fortiori it is zero in $\mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n)(x_C)$. Now a general section $\alpha \in \mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n)(U)$ is Nisnevich locally a sum of elements $a_1 \otimes \ldots \otimes a_n$ as above and hence it admits a modulus with support equal to $C \setminus U$ by Theorem 1.4.8. Thus $\mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n)$ is a reciprocity functor.

Next we claim that the n-linear morphism in **PT** (4.5) indeed induces an n-linear morphism of reciprocity functors τ. We have to check that (L3) holds. For this take effective divisors $\mathfrak{m}_i$ on C and elements $a_i \in \mathscr{M}_i(C, \mathfrak{m}_i)$. As we saw above this gives in $\mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n)(x_C)$

$$\begin{aligned}\sum_{Q \in |\max_i\{\mathfrak{m}_i\}|} & (\tau(a_1, \ldots, a_n), f)_Q = \\ & - \sum_{Q \in C \setminus |\max_i\{\mathfrak{m}_i\}|} v_Q(f) \mathrm{Tr}_{Q/x_C} s_Q(\tau(a_1, \ldots, a_n)) = 0\end{aligned}$$

for all $f \equiv 1 \bmod \max_i\{\mathfrak{m}_i\}$. Hence

$$\tau(a_1, \ldots, a_n) \in \mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n)(C, \max_i\{\mathfrak{m}_i\}).$$

By Remark 4.2.2 this gives (L3).

The n-linear map (4.6) induces a natural transformation of functors $\mathbf{RF} \to (R - \mathrm{mod})$

$$\mathrm{Hom}_{\mathbf{RF}}(\mathrm{T}(\mathscr{M}_1, \ldots \mathscr{M}_n), -) \to n - \mathrm{Lin}(\mathscr{M}_1, \ldots, \mathscr{M}_n; -)$$

and we claim, that it is in fact an isomorphism of functors. For this it suffices to show that it is an isomorphism of R-modules when evaluated at any $\mathscr{N} \in \mathbf{RF}$. The injectivity follows immediately from the fact that for a regular connected scheme $X \in \mathrm{Reg}^{\leqslant 1}$ any element in $\mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n)(X)$ can be written as a sum of elements of the form $\mathrm{Tr}_{Y/X}(\tau(a_1, \ldots, a_n))$, where $Y \to X$ is a finite flat morphism in $\mathrm{RegCon}^{\leqslant 1}$ and $a_i \in \mathscr{M}_i(Y)$. For the surjectivity let $\Phi : \mathscr{M}_1 \times \cdots \times \mathscr{M}_n \to \mathscr{P}$

be an n-linear morphism between reciprocity functors. Then the corresponding n-linear map on the underlying presheaves with transfers factors over $\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_n$. Furthermore since Φ satisfies (L3) we get

$$\Phi((4.4)) = \sum_{Q \in C \setminus |\max_i\{\mathfrak{m}_i\}|} v_Q(f)\mathrm{Tr}_{Q/x_C} s_Q^{\mathscr{P}}(\Phi(a_1, \dots, a_n)) = 0.$$

Thus Φ factors over $\mathrm{LT}(\mathscr{M}_1, \dots, \mathscr{M}_n)$ and we obtain a morphism of lax Mackey functors with specialization map $\mathrm{LT}(\mathscr{M}_1, \dots, \mathscr{M}_n) \to \mathscr{P}$. The left adjoint property of Σ hence gives a morphism in $\mathbf{RF}$

$$\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n) = \Sigma(\mathrm{LT}(\mathscr{M}_1, \dots, \mathscr{M}_n)) \to \mathscr{P}.$$

This finishes the proof. □

4.2.5. **Corollary.** *Let $\mathscr{M}_i$, $\mathscr{M}_i'$, $i = 1, \dots, n$, be reciprocity functors.*

(1) For all $1 \leqslant i < j \leqslant n$ we have a functorial isomorphism

$$\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_i, \dots, \mathscr{M}_j, \dots, \mathscr{M}_n) \cong \mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_j, \dots, \mathscr{M}_i, \dots, \mathscr{M}_n).$$

(2) For all $1 \leqslant i \leqslant n$ we have a functorial isomorphism

$$\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_i \oplus \mathscr{M}_i', \dots, \mathscr{M}_n) \\ \cong \mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_i \dots, \mathscr{M}_n) \oplus \mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_i' \dots, \mathscr{M}_n).$$

(3) There is a functorial map

$$\mathrm{T}(\mathscr{M}_1, \mathscr{M}_2, \mathscr{M}_3) \to \mathrm{T}(\mathrm{T}(\mathscr{M}_1, \mathscr{M}_2), \mathscr{M}_3),$$

which is surjective as a map of Nisnevich sheaves (i.e. it is a strict epimorphism in $\mathbf{RF}$, see Lemma 3.2.8).

Proof. (1) and (2) follow immediately from the universal property. The existence of the map in (3) follows from the universal property and the natural map

$$\mathrm{Bil}_{\mathbf{RF}}(\mathrm{T}(\mathscr{M}_1, \mathscr{M}_2), \mathscr{M}_3; \mathscr{N}) \to 3 - \mathrm{Lin}_{\mathbf{RF}}(\mathscr{M}_1, \mathscr{M}_2, \mathscr{M}_3; \mathscr{N}).$$

The surjectivity statement follows from the fact that both sides are quotients of $\mathscr{M}_1 \otimes \mathscr{M}_2 \otimes \mathscr{M}_3$. □

4.2.6. *Remark.* We don't know whether the map in (3) above is an isomomorphism (maybe one has to impose further conditions on the $\mathscr{M}_i$'s). Thus we don't know whether T is associative, and we cannot call it a tensor product.

4.2.7. **Corollary.** *Let $\mathscr{M}_1, \dots, \mathscr{M}_n$ be reciprocity functors and let N be an R-module viewed as a constant reciprocity functor. Then we have a functorial isomorphism of reciprocity functors*

$$\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n, N) \cong \Sigma(\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n) \otimes N),$$

where $\otimes$ on the right hand side is the tensor product in $\mathbf{PT}$ (see Example 4.1.7). In particular, $\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n, R) = \mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n)$ and for R-modules N_i we have $\mathrm{T}(N_1, \dots, N_n) = N_1 \otimes_R \dots \otimes_R N_n$.

Proof. It suffices to show that (in the notation of Definition 4.2.3) we have

$$\mathrm{LT}(\mathscr{M}_1, \dots, \mathscr{M}_n, N) = \mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n) \otimes N,$$

which is clear since (4.4) is already zero on the right hand side. □

4.2.8. **Corollary.** *For $\mathscr{M}_1, \dots, \mathscr{M}_n \in \mathbf{RF}_n$ (see Definitions 1.5.7 for the notation), we have $\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n) \in \mathbf{RF}_n$, i.e. T restricts to a functor*

$$\mathrm{T} : \prod \mathbf{RF}_n \to \mathbf{RF}_n.$$

Proof. Let $C \in (\mathscr{C}/S)$ be a curve and $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ effective divisors on it. Then by (L3) the n-linear map τ induces a map

$$\mathscr{M}_1(C, \mathfrak{m}_1) \times \ldots \times \mathscr{M}_n(C, \mathfrak{m}_n) \to \mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n)(C, \max\{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}).$$

Now the statement follows from Proposition 1.4.5, (3) and the fact that any element in $\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n)(\eta_C)$ can be written as a sum of elements of the form $\mathrm{Tr}_{\eta_D/\eta_C}(\tau(a_1, \dots, a_n))$, with $D \to C$ a finite and flat map in $(\mathscr{C}/S)$ and $a_i \in \mathscr{M}_i(\eta_D)$. □

4.2.9. **Corollary.** *Let $\mathscr{M}_1, \dots, \mathscr{M}_{n-1}$ be reciprocity functors. Then*

$$\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_{n-1}, -) : \mathbf{RF} \to \mathbf{RF}$$

is right exact in the sense of 3.2.2.

Proof. We write $\underline{\mathscr{M}} = (\mathscr{M}_1, \dots, \mathscr{M}_{n-1})$. Let

$$0 \to \mathscr{N}' \to \mathscr{N} \to \mathscr{N}'' \to 0$$

be a short exact sequence in $\mathbf{RF}$ (in the sense of 3.2.2). Then we have to show

$$\mathrm{Coker}_{\mathbf{RF}}(\mathrm{T}(\underline{\mathscr{M}}, \mathscr{N}') \to \mathrm{T}(\underline{\mathscr{M}}, \mathscr{N})) \cong \mathrm{T}(\underline{\mathscr{M}}, \mathscr{N}'').$$

But since $\mathscr{N}'' \cong \mathrm{Coker}_{\mathbf{RF}}(\mathscr{N}' \to \mathscr{N})$ can be described as in Remark 3.2.10 it is easy to check that for all $\mathscr{P} \in \mathbf{RF}$ we have an exact sequence of R-modules

$$0 \to n - \mathrm{Lin}(\underline{\mathscr{M}}, \mathscr{N}''; \mathscr{P}) \to n - \mathrm{Lin}(\underline{\mathscr{M}}, \mathscr{N}; \mathscr{P}) \to n - \mathrm{Lin}(\underline{\mathscr{M}}, \mathscr{N}'; \mathscr{P}).$$

Thus using the universal property of T we see that the natural map $T(\underline{\mathscr{M}}, \mathscr{N}) \to \mathrm{T}(\underline{\mathscr{M}}, \mathscr{N}'')$ satisfies the universal property of the cokernel. Hence the statement. □

4.2.10. *Remark.* Let $F \subset F'$ be a finite extension of perfect fields, giving rise to a map $S' = \operatorname{Spec} F' \to S = \operatorname{Spec} F$. In the following we indicate by a label S or S' with respect to which base we are working. We get a natural functor $\mathrm{Reg}^{\leqslant 1}\mathrm{Cor}_{S'} \to \mathrm{Reg}^{\leqslant 1}\mathrm{Cor}_S$, which induces a functor $\mathbf{RF}_S \to \mathbf{RF}_{S'}$. Thus given $\mathscr{M}_1, \dots, \mathscr{M}_n \in \mathbf{RF}_S$ we can view the n-linear map $\tau : \mathscr{M}_1 \times \ldots \times \mathscr{M}_n \to \mathrm{T}_S(\mathscr{M}_1, \dots, \mathscr{M}_n)$ as an n-linear map of reciprocity functors over S' and hence we get a map

$$\mathrm{T}_{S'}(\mathscr{M}_1, \dots, \mathscr{M}_n) \twoheadrightarrow \mathrm{T}_S(\mathscr{M}_1, \dots, \mathscr{M}_n), \quad \tau_{S'}(\underline{m}) \mapsto \tau_S(\underline{m}) \quad \text{in } \mathbf{RF}_{S'},$$

which is automatically surjective since both sides are quotients of $\mathscr{M}_1 \otimes \ldots \otimes \mathscr{M}_n$.

4.2.11. *Notation.* Let $\mathscr{M}$ and $\mathscr{N}$ be two reciprocity functors. Then we will use the following notation (and variants of it)

$$\mathrm{T}(\mathscr{M}, \mathscr{N}^{\times n}) := \mathrm{T}(\mathscr{M}, \underbrace{\mathscr{N}, \dots, \mathscr{N}}_{n\text{-times}}).$$

5. Computations

5.1. Relation with homotopy invariant Nisnevich sheaves with transfers.

5.1.1. *Tensor product for presheaves with transfers.* By [35, 3.2] the Abelian category **PST** (see 2.3.1) has a monoidal structure extending the one on SmCor via the Yoneda embedding. By [33, §2] it is given by the following formula: Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves with transfers, then $\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G}$ is the presheaf with transfers which on $X \in \mathrm{SmCor}$ is given by

$$(\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G})(X) = \bigoplus_{Y,Z \in Sm} \mathscr{F}(Y) \otimes_{\mathbb{Z}} \mathscr{G}(Z) \otimes_{\mathbb{Z}} \mathrm{Cor}(X, Y \times Z)/\Lambda, \tag{5.1}$$

where Sm is the category of smooth S-schemes and Λ is the subgroup generated by elements of the following form

$$\phi \otimes \psi \otimes (f \times \mathrm{id}_Z) \circ h - f^*(\phi) \otimes \psi \otimes h, \tag{5.2}$$

where $\phi \in \mathscr{F}(Y)$, $\psi \in \mathscr{G}(Z)$, $f \in \mathrm{Cor}(Y', Y)$, $h \in \mathrm{Cor}(X, Y' \times Z)$, and

$$\phi \otimes \psi \otimes (\mathrm{id}_Y \times g) \circ h - \phi \otimes g^*(\psi) \otimes h, \tag{5.3}$$

where $\phi \in \mathscr{F}(Y)$, $\psi \in \mathscr{G}(Z)$, $g \in \mathrm{Cor}(Z', Z)$, $h \in \mathrm{Cor}(X, Y \times Z')$.

5.1.2. The categories **HI**, **NST** and $\mathbf{HI}_{\mathrm{Nis}}$ inherit symmetric monoidal structures defined respectively by:

$$F \otimes_{\mathbf{NST}} G := (F \otimes_{\mathbf{PST}} G)_{\mathrm{Nis}}, \qquad F \otimes_{\mathbf{HI}} G := h_0(F \otimes_{\mathbf{PST}} G),$$
$$F \otimes_{\mathbf{HI}_{\mathrm{Nis}}} G := h_0^{\mathrm{Nis}}(F \otimes_{\mathbf{PST}} G),$$

where $h_0 : \mathbf{PST} \to \mathbf{HI}$ is the left adjoint of the forgetful functor $\mathbf{HI} \to \mathbf{PST}$ (see 2.3.1) and h_0^{Nis} is the composition of h_0 with the sheafification functor.

5.1.3. **Proposition.** *Let $\mathscr{F}, \mathscr{G} \in$ **PST** be two presheaves with transfers. There exists a canonical and functorial isomorphism in* **PT**

$$(\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G})^{\frown} \cong \hat{\mathscr{F}} \otimes \hat{\mathscr{G}},$$

*where $(-)^{\frown} :$ **PST** $\to$ **PT** is the functor from Proposition 2.3.4 and the tensor product on the right hand side is the one defiend in 4.1.1.*

Proof. Let us first construct a morphism

$$\theta : (\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G})^{\frown} \to \hat{\mathscr{F}} \otimes \hat{\mathscr{G}} \quad \text{in } \mathbf{PT}.$$

Let X be a scheme in $\mathrm{Reg}^{\leqslant 1}$. Given a model $U \in \mathfrak{M}_X$ of X, we define a map

$$\theta_{X,U} : \bigoplus_{Y,Z \in \mathrm{Sm}} \mathscr{F}(Y) \otimes_{\mathbb{Z}} \mathscr{G}(Z) \otimes_{\mathbb{Z}} \mathrm{Cor}(U, Y \times Z) \to (\hat{\mathscr{F}} \otimes \hat{\mathscr{G}})(X) \tag{5.4}$$

as follows: For $\alpha \in \mathscr{F}(Y) \otimes \mathscr{G}(Z)$ with $Y, Z \in \mathrm{Sm}$ and $[W] \in \mathrm{Cor}(U, Y \times Z)$ an elementary correspondence, take an open subset $U' \subset U$ which is a model of X and such that the normalization $\tilde{W}'$ of the pullback of W along U' is smooth (this is possible since the image of X in U is contained is the locus of at most 1-codimensional points); then we define

$$\theta_{X,U}(\alpha \otimes [W]) := \text{class of } (p^*_{\tilde{W}',Y} \otimes p^*_{\tilde{W}',Z})(\alpha) \text{ in } (\hat{\mathscr{F}} \otimes \hat{\mathscr{G}})(X),$$

where $p_{\tilde{W}',Y} : \tilde{W} \to Y$ and $p_{\tilde{W}',Z} : \tilde{W} \to Z$ are induced by projection. Clearly this element is independent of the choice of U' and we thus obtain a well-defined map (5.4). We claim that $\theta_{X,U}$ factors to give a map (again denoted by $\theta_{X,U}$)

$$\theta_{X,U} : (\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G})(U) \to (\hat{\mathscr{F}} \otimes \hat{\mathscr{G}})(X).$$

For this we have to show that $\theta_{X,U}$ sends the elements (5.2) and (5.3) to zero. Notice that we can assume that the correspondences f, g, h appearing in these formulas are elementary. So take $Y, Y', Z \in \mathrm{Sm}$ and $\phi \in \mathscr{F}(Y)$, $\psi \in \mathscr{G}(Z)$ and elementary correspondences $f \in \mathrm{Cor}(Y', Y)$, $h \in \mathrm{Cor}(U, Y' \times Z)$ as in (5.2). After replacing U with a smaller model of X, we can assume that $\tilde{h}$ - the normalization of h - is

smooth and further that the normalizations of all irreducible components of the scheme-theoretic intersection of

$$I := (h \times Y \times Z) \cap (U \times (f \times \mathrm{id}_Z))$$

inside $U \times Y' \times Z \times Y \times Z$ are smooth as well as the normalization of their (reduced) images in $U \times Y \times Z$. (We can do this since all this schemes are finite and surjective over U.) For an irreducible component T of I set

$$S_T = p_{UYZ}(T)_{\mathrm{red}} \subset U \times Y \times Z$$

and denote by $d_T := [T : S_T]$ the degree of T over S_T and by n_T the intersection number of T inside I. Thus

$$[h \times Y \times Z] \cdot [U \times (f \times \mathrm{id}_Z)] = \sum_{T \subseteq I} n_T [T]$$

and by definition

$$(f \times \mathrm{id}_Z) \circ h = \sum_{T \subseteq I} n_T \cdot d_T \cdot [S_T] \in \mathrm{Cor}(U, Y \times Z).$$

We introduce some more notations using the following commutative diagram in which all maps are the natural once (induced by composition of embeddings, projections and normalization):

$$\begin{tikzcd}
 & \tilde{S}_T \arrow[r, "p_{S,Y}"] \arrow[rrd, bend left, "p_{S,Z}"] & Y & \\
 & \tilde{T} \arrow[u, "\pi_T"] \arrow[ur, "p_{T,Y}"'] \arrow[r] \arrow[d, "p_{T,h}"'] & f \times \mathrm{id}_Z \arrow[u] \arrow[r] \arrow[d] & Z \\
Y' & \tilde{h} \arrow[l, "p_{h,Y'}"] \arrow[r] \arrow[rru, bend right, "p_{h,Z}"'] & Y' \times Z. \arrow[ur] &
\end{tikzcd} \tag{5.5}$$

Thus we get in $(\hat{\mathscr{F}} \otimes \hat{\mathscr{G}})(X)$ (where 'p.f.' refers to the projection formula coming from the definition of $\mathcal{R}$ in 4.1.1)

$$\begin{aligned}
\theta_{X,U}(\phi \otimes \psi \otimes (f \times \mathrm{id}_Y) \circ h) &= \sum_{T \subseteq I} n_T \cdot d_T \cdot (p_{S,Y}^* \phi \otimes p_{S,Z}^* \psi), && \text{by def. of } \theta_{X,U}, \\
&= \sum_{T \subseteq I} n_T \cdot (\pi_{T*} \pi_T^* p_{S,Y}^* \phi) \otimes p_{S,Z}^* \psi \\
&= \sum_{T \subseteq I} n_T \cdot p_{T,Y}^* \phi \otimes p_{T,h}^* p_{h,Z}^* \psi, && \text{by p.f. and (5.5)} \\
&= \sum_{T \subseteq I} n_T \cdot p_{T,h*} p_{T,Y}^* \phi \otimes p_{h,Z}^* \psi, && \text{by p.f.}
\end{aligned}$$

Now observe, that we have a canonical isomorphism

$$I \cong (h \times Y \times Z) \times_{U \times Y' \times Z \times Y \times Z} (U \times f \times \mathrm{id}_Z) \cong h \times_{Y'} f.$$

Furthermore, the map $\tilde{h} \times_{Y'} f \to h \times_{Y'} f$ is an isomorphism over an open and dense subset of h and thus induces finite birational maps $T' \to T$ between the irreducible components; hence their normalizations are equal, $\tilde{T}' = \tilde{T}$. For T' an irreducible

component of $\tilde{h} \times_{Y'} f$ denote by $S_{T'}$ its reduced image (via projection) in $\tilde{h} \times Y$ and by $\tilde{S}_{T'}$ its normalization. We obtain a commutative diagram

$$\begin{tikzcd}
 & & Y \\
\tilde{T} = \tilde{T}' \arrow[r, "\rho_T"] \arrow[urr, bend left, "p_{T,Y}"] \arrow[drr, bend right, "p_{T,h}"'] & \tilde{S}_{T'} \arrow[r, hook] \arrow[ur, bend left, "p_{S_{T'},Y}"'] \arrow[dr, bend right, "p_{S_{T'},h}"] & \tilde{h} \times Y \arrow[u] \arrow[d] \\
 & & \tilde{h}.
\end{tikzcd}$$

Further if T is an irreducible component in I with intersection multiplicity n_T and T' is the corresponding irreducible component in $\tilde{h} \times_{Y'} f$, then (we denote by η_T the generic point of T and by abuse of notation also its image in the various other schemes appearing)

$$\begin{aligned}
n_T &= \sum_{i \geqslant 0} (-1)^i \mathrm{lg}\left(\mathrm{Tor}_i^{\mathscr{O}_{Y' \times Z, \eta_T}}(\mathscr{O}_{h,\eta_T}, \mathscr{O}_{f \times \mathrm{id}_Z, \eta_T})\right) \\
&= \sum_{i \geqslant 0} (-1)^i \mathrm{lg}\left(\mathrm{Tor}_i^{\mathscr{O}_{Y', \eta_T}}(\mathscr{O}_{\tilde{h},\eta_T}, \mathscr{O}_{f, \eta_T})\right),
\end{aligned}$$

which is the intersection number of T' in $\tilde{h} \times_{Y'} f$. Thus

$$f \circ [\Gamma_{p_{h,Y'}}] = \sum_{T \subset I} n_T \cdot \deg \rho_T \cdot [\tilde{S}_{T'}].$$

We get

$$\begin{aligned}
\theta_{X,U}(\phi \otimes \psi \otimes (f \times \mathrm{id}_Y) \circ h) &= \sum_{T \subseteq I} n_T \cdot p_{T,h*} p_{T,Y}^* \phi \otimes p_{h,Z}^* \psi \\
&= \sum_{T \subseteq I} n_T \cdot \deg \rho_T \cdot p_{S_{T'},h*} p_{S_{T'},Y}^* \phi \otimes p_{h,Z}^* \psi \\
&= (f \circ [\Gamma_{p_{h,Y'}}])^* \phi \otimes p_{h,Z}^* \psi \\
&= p_{h,Y'}^* f^* \phi \otimes p_{h,Z}^* \psi \\
&= \theta_{X,U}(f^* \phi \otimes \psi \otimes h).
\end{aligned}$$

Hence $\theta_{X,U}((5.2)) = 0$ and similarly one checks $\theta_{X,U}((5.3)) = 0$. We thus obtain a well-defined map

$$\theta_X : (\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G})^{\frown}(X) \to (\hat{\mathscr{F}} \otimes \hat{\mathscr{G}})(X). \tag{5.6}$$

It is straightforward to check that the θ_X, $X \in \mathrm{Reg}^{\leqslant 1}$, induce a morphism of presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$

$$\theta : (\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G})^{\frown} \to (\hat{\mathscr{F}} \otimes \hat{\mathscr{G}}),$$

which clearly is functorial in $\mathscr{F}$ and $\mathscr{G}$. Thus it remains to show that (5.6) is an isomorphism for all X. We will give the inverse map. Let X be in $\mathrm{RegCon}^{\leqslant 1}$ and define a map

$$\nu_X : \hat{\mathscr{F}}(X) \times \hat{\mathscr{G}}(X) \to (F \otimes_{\mathbf{PST}} \mathscr{G})^{\frown}(X)$$

as follows. For $(\hat{a}, \hat{b})$ in $\hat{\mathscr{F}}(X) \times \hat{\mathscr{G}}(X)$ take a model $U \in \mathfrak{M}_X$ so that there are elements $a \in \mathscr{F}(U)$ and $b \in \mathscr{G}(U)$ representing $\hat{a}$ and $\hat{b}$, respectively. Then define

$$\nu_X(\hat{a}, \hat{b}) := \text{class of } a \otimes b \otimes [\Gamma_{\delta_U}] \text{ in } (\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G})^{\frown}(X),$$

where $\delta_U : U \to U \times U$ is the diagonal morphism and $[\Gamma_{\delta_U}] \in \mathrm{Cor}(U, U \times U)$ is its graph. Clearly ν_X does not depend on the choice of U or of the representatives a

and b. Varying X we claim that the ν_X's define a bilinear map of presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$ (see Definition 4.1.5)

$$\nu : \hat{\mathscr{F}} \times \hat{\mathscr{G}} \to (\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G})\widehat{}. \tag{5.7}$$

Indeed the property (L1) follows immediately from

$$\delta_U \circ \varphi = (\varphi \times \varphi) \circ \delta_V = (\varphi \times \mathrm{id}_U) \circ (\mathrm{id}_V \times \varphi) \circ \delta_V$$

for each map $\varphi : V \to U$ in Sm. The property (L2) follows from the following equality in $(\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G})(U)$ which holds for all finite maps $\varphi : V \to U$ in Sm and all $a \in \mathscr{F}(V)$ and $b \in \mathscr{G}(U)$ (we denote by Γ_φ the graph of φ and by Γ_φ^t its transpose):

$$\begin{aligned}
(\varphi_* a) \otimes b \otimes [\Gamma_{\delta_U}] &= ([\Gamma_\varphi^t]^* a) \otimes b \otimes [\Gamma_{\delta_U}] \\
&= a \otimes b \otimes (([\Gamma_\varphi^t] \times \mathrm{id}_U) \circ [\Gamma_{\delta_U}]), && \text{by (5.2)}, \\
&= a \otimes b \otimes ((\mathrm{id}_V \times [\Gamma_\varphi]) \circ [\Gamma_{\delta_V}] \circ [\Gamma_\varphi^t]) \\
&= a \otimes \varphi^* b \otimes ([\Gamma_{\delta_V}] \circ [\Gamma_\varphi^t]), && \text{by (5.3)}.
\end{aligned}$$

Thus ν is a bilinear map of presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$ and hence induces a morphism presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$ (also denoted by ν)

$$\nu : \hat{\mathscr{F}} \otimes \hat{\mathscr{G}} \to (\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G})\widehat{}.$$

It is immediate that $\theta \circ \nu = \mathrm{id}$. The equality $\nu \circ \theta = \mathrm{id}$ follows from the following equality in $(\mathscr{F} \otimes_{\mathbf{PST}} \mathscr{G})(U)$, which holds for all $Y, Z \in \mathrm{Sm}$ and all $a \in \mathscr{F}(Y)$, $b \in \mathscr{G}(Z)$ and all elementary correspondences $[W] \in \mathrm{Cor}(U, X \times Y)$ such that $\tilde{W}$ is smooth (where we denote by p_U, p_X and p_Y the maps induced by the respective projections from $\tilde{W}$):

$$\begin{aligned}
p_Y^* a \otimes p_Z^* b \otimes ([\Gamma_{\delta_{\tilde{W}}}] \circ [\Gamma_{p_U}^t]) &= a \otimes p_Z^* b \otimes (([\Gamma_{p_Y}] \times \mathrm{id}_{\tilde{W}}) \circ [\Gamma_{\delta_{\tilde{W}}}] \circ [\Gamma_{p_U}^t]) \\
&= a \otimes b \otimes ((\mathrm{id}_Y \times \Gamma_{p_Z}) \circ ([\Gamma_{p_Y}] \times \mathrm{id}_{\tilde{W}}) \circ [\Gamma_{\delta_{\tilde{W}}}] \circ [\Gamma_{p_U}^t]) \\
&= a \otimes b \otimes [W].
\end{aligned}$$

This finishes the proof. □

5.1.4. **Definition.** Let $\mathscr{M} \in \mathbf{LMFsp}$ be a lax Mackey functor with specialization map and x an S-point. One defines the R-module $\mathrm{K}^{\mathrm{geo}}(x; \mathscr{M})$ as the quotient in the category of R-modules of $\mathscr{M}(x)$ by the submodule generated by the elements

$$\sum_{P \in U} v_P(f) \cdot \mathrm{Tr}_{P/x}(s_P^{\mathscr{M}}(a))$$

where $C \in (\mathscr{C}/S)$ is a curve of finite type over x, $U \subseteq C$ is an open subset, $f \in \mathbb{G}_m(\eta_C)$ with $f \in U_P^{(1)}$, for all $P \in C \setminus U$, and $a \in \mathscr{M}(U)$.

Note that this kind of quotient was already considered in [16, 6.1 Definition]. By construction we have canonical surjections

$$\mathscr{M}(x) \twoheadrightarrow \mathrm{K}^{\mathrm{geo}}(x; \mathscr{M}) \twoheadrightarrow \mathrm{L}\varrho_1(\mathscr{M})(x) \twoheadrightarrow \varrho_1(\mathscr{M})(x).$$

5.1.5. Let $\mathscr{F} \in \mathbf{PST}$ be a presheaf with transfers. Applying the functor $\widehat{\ }$ from Proposition 2.3.4 to the canonical morphism of presheaves with transfers $a : \mathscr{F} \to h_0^{\mathrm{Nis}}(\mathscr{F})$ provides a morphism

$$\hat{a} : \widehat{\mathscr{F}} \to \widehat{h_0^{\mathrm{Nis}}(\mathscr{F})}$$

in **LMFsp**. By Proposition 2.3.5, the right hand side is a reciprocity functor that lies in $\mathbf{RF}_1$, therefore using the adjunction from Lemma 3.3.1, we obtain a morphism of reciprocity functors

$$\hat{a}^\sharp : \varrho_1(\widehat{\mathscr{F}}) \to \widehat{h_0^{\mathrm{Nis}}(\mathscr{F})}. \tag{5.8}$$

The following proposition may be seen as a generalization of [16, 6.5 Proposition].

5.1.6. **Proposition.** *Let $\mathscr{F} \in$ **PST** be a presheaf with transfers. Then the morphism (5.8) is an isomorphism of reciprocity functors. Moreover for any S-point x, the morphisms*

$$\mathrm{K}^{\mathrm{geo}}(x;\hat{\mathscr{F}}) \twoheadrightarrow \varrho_1(\hat{\mathscr{F}})(x) \xrightarrow{\hat{a}^\sharp} \widehat{h_0^{\mathrm{Nis}}(\mathscr{F})}(x),$$

where $\mathrm{K}^{\mathrm{geo}}(x;\hat{\mathscr{F}})$ is the $\mathbb{Z}$-module defined in 5.1.4, are isomorphisms.

Proof. Since $a : \mathscr{F} \to h_0^{\mathrm{Nis}}(\mathscr{F})$ is a surjection on Nisnevich stalks, so is $\hat{a}$. Further by the construction of ϱ_1 in Lemma 3.3.1, there is a natural map $\hat{\mathscr{F}} \to \varrho_1(\hat{\mathscr{F}})$ whose composition with $\hat{a}^\sharp$ is the map $\hat{a}$. Thus $\hat{a}^\sharp$ is a surjection of Nisnevich sheaves. By the conditions (Inj) and (F.P.) which a reciprocity functor satisfies it suffices to check the injectivity on S-points. Thus it suffices to show that the surjective composition from the statement

$$b : \mathrm{K}^{\mathrm{geo}}(x;\hat{\mathscr{F}}) \twoheadrightarrow \widehat{h_0^{\mathrm{Nis}}(\mathscr{F})}(x)$$

is injective. Notice that for an S-point x the $\mathbb{Z}$-module $\widehat{h_0^{\mathrm{Nis}}(\mathscr{F})}(x)$ is just the generic stalk of $h_0^{\mathrm{Nis}}(\mathscr{F})$ viewed as a Nisnevich on some model of x; in particular

$$\widehat{h_0^{\mathrm{Nis}}(\mathscr{F})}(x) = \widehat{h_0(\mathscr{F})}(x) = \mathrm{Coker}\left[\hat{\mathscr{F}}_{\mathbb{P}^1_x}(\mathbb{A}^1_x) \xrightarrow{i_0^*-i_1^*} \hat{\mathscr{F}}(x)\right].$$

But now consider on $\mathbb{P}^1_x$ the modulus $\mathfrak{m} = \{\infty\}$ and the rational function $f = \frac{t}{t-1} \in \kappa(x)(t)^\times$ which is congruent to one modulo $\mathfrak{m}$. Given $\alpha \in \hat{\mathscr{F}}_{\mathbb{P}^1_x}(\mathbb{A}^1_x)$, we have

$$i_0^*(\alpha) - i_1^*(\alpha) = \sum_{P \in \mathbb{A}^1_x} v_P(f)\mathrm{Tr}_{P/x}(s_P^{\hat{\mathscr{F}}}(\alpha))$$

in $\hat{\mathscr{F}}(x)$ and by definition this element vanishes in $\mathrm{K}^{\mathrm{geo}}(x;\hat{\mathscr{F}})$. Thus the natural surjection $\hat{\mathscr{F}}(x) \to \mathrm{K}^{\mathrm{geo}}(x;\hat{\mathscr{F}})$ factors via $\widehat{h_0^{\mathrm{Nis}}(\mathscr{F})}(x)$ and gives an inverse of b. This proves the statement. □

5.1.7. **Lemma.** *Let $\mathscr{F}_1, \dots, \mathscr{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. Then*

$$\varrho_1(\hat{\mathscr{F}}_1 \otimes \cdots \otimes \hat{\mathscr{F}}_n) = \mathrm{T}(\hat{\mathscr{F}}_1, \cdots, \hat{\mathscr{F}}_n) \quad \text{in } \mathbf{RF}_1.$$

Proof. By Corollary 4.2.8 $\mathrm{T}(\hat{\mathscr{F}}_1, \dots, \hat{\mathscr{F}}_n) \in \mathbf{RF}_1$. Thus applying ϱ_1 to the natural map $\hat{\mathscr{F}}_1 \otimes \ldots \otimes \hat{\mathscr{F}}_n \to \mathrm{T}(\hat{\mathscr{F}}_1, \dots, \hat{\mathscr{F}}_n)$ gives a canonical map

$$\varrho_1(\hat{\mathscr{F}}_1 \otimes \cdots \otimes \hat{\mathscr{F}}_n) \to \mathrm{T}(\hat{\mathscr{F}}_1, \cdots, \hat{\mathscr{F}}_n). \tag{5.9}$$

On the other hand we have a natural n-linear map of presheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$

$$\hat{\mathscr{F}}_1 \times \ldots \times \hat{\mathscr{F}}_n \to \varrho_1(\hat{\mathscr{F}}_1 \otimes \cdots \otimes \hat{\mathscr{F}}_n),$$

which automatically satisfies (L3) (see Definition 4.2.1), since the right hand side is in $\mathbf{RF}_1$. Thus it is an n-linear map of reciprocity functors and hence induces an inverse to (5.9). □

5.1.8. **Theorem.** *Let $\mathscr{F}_1, \dots, \mathscr{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$ be homotopy invariant Nisnevich sheaves with transfers. There exists a canonical and functorial isomorphism of reciprocity functors*

$$\mathrm{T}(\hat{\mathscr{F}}_1, \dots, \hat{\mathscr{F}}_n) \xrightarrow{\sim} (\mathscr{F}_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathscr{F}_n)\widehat{\ }. \tag{5.10}$$

Moreover for any S-point x the canonical morphism

$$\mathrm{K}^{\mathrm{geo}}(x, \hat{\mathscr{F}}_1 \otimes \ldots \otimes \hat{\mathscr{F}}_n) \to \mathrm{T}(\hat{\mathscr{F}}_1, \dots, \hat{\mathscr{F}}_n)(x)$$

is an isomorphism.

Proof. Let $\mathscr{F} := \mathscr{F}_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} \mathscr{F}_n$ and $a : \mathscr{F} \to h_0^{\mathrm{Nis}}(\mathscr{F})$ be the canonical morphism of presheaves with transfers. By Proposition 5.1.3, we have a canonical and functorial isomorphism in **LMFsp**

$$\hat{\mathscr{F}}_1 \otimes \cdots \otimes \hat{\mathscr{F}}_n \xrightarrow{\nu} \hat{\mathscr{F}}.$$

Thus by Proposition 5.1.6 we get an isomorphism of reciprocity functors

$$\varrho_1\left(\hat{\mathscr{F}}_1 \otimes \cdots \otimes \hat{\mathscr{F}}_n\right) \xrightarrow{\varrho_1(\nu)} \varrho_1(\hat{\mathscr{F}}) \xrightarrow{\hat{a}^\sharp} \left[h_0^{\mathrm{Nis}}(\mathscr{F})\right]\widehat{\ }.$$

Since $h_0^{\mathrm{Nis}}(\mathscr{F}) = \mathscr{F}_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathscr{F}_n$ by definition the first statement follows from Lemma 5.1.7. The second part of the statement is an immediate consequence of Proposition 5.1.6 and Lemma 5.1.7. □

Using the main result of [16] we get:

5.1.9. **Corollary.** *Let $\mathscr{F}_1, \dots, \mathscr{F}_n$ be homotopy invariant Nisnevich sheaves with transfers on Sm_S with $S = \operatorname{Spec} F$ the spectrum of a perfect field. Denote by*

$$\mathrm{K}(F, \mathscr{F}_1, \dots, \mathscr{F}_n)$$

the K-group defined in [16, Def. 5.1] *(in particular if the $\mathscr{F}_i$'s are semi-Abelian varieties, it coincides with Somekawa's K-group defined in* [32, 1.]*). Then there is an isomorphism*

$$\mathrm{T}(S, \hat{\mathscr{F}}_1, \dots, \hat{\mathscr{F}}_n) \cong \mathrm{K}(F, \mathscr{F}_1, \dots, \mathscr{F}_n).$$

Proof. This follows from Theorem 5.1.8 together with the fact that $\mathrm{K}^{\mathrm{geo}}(S, \hat{\mathscr{F}}_1 \otimes \ldots \otimes \hat{\mathscr{F}}_n)$ coincides by its very definition with the K-group of geometric type $\mathrm{K}'(F, \mathscr{F}_1, \dots, \mathscr{F}_n)$ defined in [16, Def. 6.1] and which by [16, Thm 6.2] and [16, Thm. 11.12] is isomorphic to $\mathrm{K}(F, \mathscr{F}_1, \dots, \mathscr{F}_n)$. □

5.2. **Applications.**

5.2.1. We now relate, as in [16], the K-groups of some reciprocity functors associated to homotopy invariant Nisnevich sheaves with transfers to Hom groups in V. Voevodsky's triangulated category of effective motivic complexes $\mathbf{DM}_-^{\mathrm{eff}}$. The main result is Theorem 5.2.3. Recall that $\mathbf{DM}_-^{\mathrm{eff}}$ is the full triangulated subcategory of the derived category $\mathrm{D}^-(\mathbf{NST})$ formed by the complexes whose cohomology sheaves are homotopy invariant. If $\mathscr{C} \in \mathbf{DM}_-^{\mathrm{eff}}$ and X is a smooth scheme of finite type over S, then the conjunction of [35, Proposition 3.1.9] and [35, Proposition 3.2.3] implies that

$$\mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(M(X), \mathscr{C}[i]) \simeq \mathbb{H}^i_{\mathrm{Nis}}(X, \mathscr{C}) \tag{5.11}$$

where $M(X) := C_*(L(X))$ is the motive of X in $\mathbf{DM}_-^{\mathrm{eff}}$ (here as usual C_* is the Suslin complex *i.e.* the $\mathbb{A}^1$-localization functor and $L(X)$ is the sheaf with transfers represented by X). For any $\mathscr{C} \in \mathbf{DM}_-^{\mathrm{eff}}$ we have

$$\operatorname*{colim}_{U \in \mathfrak{M}_x^{\mathrm{op}}} \mathbb{H}^0_{\mathrm{Nis}}(U, \mathscr{C}) \simeq \widehat{\mathsf{H}^0(\mathscr{C})}(x). \tag{5.12}$$

where $\mathsf{H}^0 : \mathrm{D}^-(\mathbf{NST}) \to \mathbf{NST}$ is the cohomological functor associated to the usual t-structure. (Since $\widehat{\mathsf{H}^0(\mathscr{C})}(x)$ is just the generic stalk of $\mathsf{H}^0(\mathscr{C})$ viewed as a Nisnevich sheaf on some model of x.)

5.2.2. Let us denote by $\otimes_{\mathbf{DM}}$ the tensor product in the category $\mathbf{DM}^{\mathrm{eff}}_-$ defined by

$$\mathscr{C} \otimes_{\mathbf{DM}} \mathscr{D} := C_*(\mathscr{C} \otimes^{\mathrm{L}} \mathscr{D})$$

where $- \otimes^{\mathrm{L}} -$ is the derived tensor product on $\mathrm{D}^-(\mathbf{NST})$ defined using "free resolutions" as in [33, §2]. In particular if $\mathscr{F}_1, \dots, \mathscr{F}_n$ are Nisnevich sheaves with transfers, then unfolding the definitions we get

$$\begin{aligned} \mathsf{H}^0(C_*(\mathscr{F}_1) \otimes_{\mathbf{DM}} \dots \otimes_{\mathbf{DM}} C_*(\mathscr{F}_n)) &= \mathsf{H}^0(C_*(C_*(\mathscr{F}_1) \otimes^L \dots \otimes^L C_*(\mathscr{F}_n))) \\ &= h_0^{\mathrm{Nis}}(h_0^{\mathrm{Nis}}(\mathscr{F}_1) \otimes_{\mathbf{NST}} \dots \otimes_{\mathbf{NST}} h_0^{\mathrm{Nis}}(\mathscr{F}_n)) \\ &= h_0^{\mathrm{Nis}}(\mathscr{F}_1) \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \dots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} h_0^{\mathrm{Nis}}(\mathscr{F}_n). \end{aligned} \tag{5.13}$$

In the sequel we use the following notations

$$M(X) := C_*(L(X)), \qquad M^c(X) := C_*(L^c(X))$$

and

$$h_0^{\mathrm{Nis}}(X) := \mathsf{H}^0(C_*(L(X))), \qquad h_0^{\mathrm{Nis},c}(X) := \mathsf{H}^0(C_*(L^c(X))),$$

where $L^c(X)$ is the presheaf with transfers such that $L^c(X)(Y)$, for $Y \in \mathrm{Sm}$, is the free Abelian group generated by the closed integral subschemes of $Y \times X$ that are quasi-finite and dominate an irreducible component of Y (see [35, §4.1]).

5.2.3. **Corollary.** *Let $X_1, \dots, X_n$ be smooth schemes of finite type over S and $r \geqslant 0$ be an integer. Set $X := X_1 \times \dots \times X_n$. Then we have isomorphisms of reciprocity functors*

$$\mathrm{T}(\widehat{h}_0^{\mathrm{Nis}}(X_1), \dots, \widehat{h}_0^{\mathrm{Nis}}(X_n), \mathbb{G}_m^{\times r}) \cong \mathsf{H}^0(M(X)(r)[r])^{\widehat{}}$$
$$\mathrm{T}(\widehat{h}_0^{\mathrm{Nis},c}(X_1), \dots, \widehat{h}_0^{\mathrm{Nis},c}(X_n), \mathbb{G}_m^{\times r}) \cong \mathsf{H}^0(M^c(X)(r)[r])^{\widehat{}}.$$

In particular by (5.11), (5.12) *we have for any S-point x and $* \in \{\emptyset, c\}$*

$$\mathrm{T}(\widehat{h}_0^{\mathrm{Nis},*}(X_1), \dots, \widehat{h}_0^{\mathrm{Nis},*}(X_n), \mathbb{G}_m^{\times r})(x) \cong \operatorname*{colim}_{U \in \mathfrak{M}_x^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_-}(M(U), M^*(X)(r)[r]).$$

Proof. Since $\mathbb{G}_m \simeq \mathbb{Z}(1)[1]$ in $\mathbf{DM}^{\mathrm{eff}}_-$, we have in $\mathbf{DM}^{\mathrm{eff}}_-$ (see [35, Proposition 4.1.7])

$$M^*(X)(r)[r] \cong M^*(X_1) \otimes_{\mathbf{DM}} \dots \otimes_{\mathbf{DM}} M^*(X_n) \otimes_{\mathbf{DM}} \mathbb{G}_m^{\otimes r},$$

with $* \in \{c, \emptyset\}$. Thus by (5.13) we get

$$\mathsf{H}^0(M^*(X)(r)[r])^{\widehat{}} = (h_0^{\mathrm{Nis},*}(X_1) \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \dots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} h_0^{\mathrm{Nis},*}(X_n) \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbb{G}_m^{\otimes r})^{\widehat{}}.$$

Hence the corollary follows from Theorem 5.1.8. $\square$

5.2.4. **Lemma.** *(1) Assume X is a smooth projective S-scheme. Then for all $r \geqslant 0$ we have an isomorphism of reciprocity functors*

$$\mathsf{H}^0(M(X)(r)[r])^{\widehat{}} \cong \mathscr{C}H_0(X, r),$$

where $\mathscr{C}H_0(X, r)$ is the reciprocity functor defined in 2.4.7.

(2) Assume X is smooth and quasi-projective and that S admits resolution of singularities. Then for all $r \geqslant 0$ we have an isomorphism of reciprocity functors

$$\mathsf{H}^0(M^c(X)(r)[r])^{\widehat{}} \cong \mathscr{C}H_0(X, r).$$

Proof. We can assume that X has equidimension d. Let $\mathrm{CH}^{d+r}_{\mathrm{Nis}}(X,r) \in \mathbf{HI}_{\mathrm{Nis}}$ be the homotopy invariant Nisnevich with transfers defined in 2.4.9. Then by (2.9) it suffices to show that we have $\mathsf{H}^0(M(X)(r)[r]) \cong \mathrm{CH}^{d+r}_{\mathrm{Nis}}(X,r)$ in the first case and $\mathsf{H}^0(M^c(X)(r)[r]) \cong \mathrm{CH}^{d+r}_{\mathrm{Nis}}(X,r)$ in the second case. Recall that the functor $\mathbf{DM}^{\mathrm{eff}}_{-} \to \mathbf{DM}_{-}$ is fully faithful by the cancellation theorem, see [37]. Further, if X is smooth projective, the motive $M(X)$ admits a dual in $\mathbf{DM}_{-}$ given by $M(X)(-d)[-2d]$ (see *e.g.* [21, Part I; Chapter IV, 1.4.2 Theorem]). Thus by [12, Corollary B.2] we have for X projective

$$M(X)(r)[r] \cong M(X)^{\vee}(d+r)[2d+r] \cong \underline{\mathrm{Hom}}(M(X), \mathbb{Z}(d+r)[2d+r])$$

and by [35, Proof of Thm 4.3.7]

$$M^c(X)(r)[r] \cong \underline{\mathrm{Hom}}(M(X), \mathbb{Z}(d+r)[2d+r])$$

in case S admits resolutions of singularities, where $\underline{\mathrm{Hom}}$ is the partially defined internal Hom in $\mathbf{DM}^{\mathrm{eff}}_{-}$. Therefore by the comparison between motivic cohomology and Higher Chow groups (see [36, Cor. 2]) we get isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}(M(U), M^*(X)(r)[r]) &\simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}(M(U\times X), \mathbb{Z}(r+d)[r+2d]) \\ &\simeq \mathrm{CH}^{d+r}(X\times U, r),\end{aligned}$$

where $* \in \{\emptyset, c\}$ depending whether we are in the first or second case. Letting U vary we get an isomorphism of presheaves with transfers, which when Nisnevich sheafified gives the claimed isomorphisms. □

Thus we can rewrite Corollary 5.2.3 as follows:

5.2.5. **Corollary.** *Let $X_1, \dots, X_n$ be smooth and quasi-projective schemes over S and $r \geqslant 0$ an integer. Assume that either the X_i's are projective or S admits resolution of singularities. Then we have an isomorphism of reciprocity functors*

$$\mathrm{T}(\mathscr{C}H_0(X_1), \dots, \mathscr{C}H_0(X_n), \mathbb{G}_m^{\times r}) \cong \mathscr{C}H_0(X_1 \times \dots \times X_n, r).$$

In particular for all S-points x we have

$$\mathrm{T}(\mathscr{C}H_0(X_1), \dots, \mathscr{C}H_0(X_n), \mathbb{G}_m^{\times r})(x) \cong \mathrm{CH}_0(X_{1,x} \times_x \dots \times_x X_{n,x}, r).$$

5.2.6. *Remark.* (1) The above corollary should be compared to the results [26, Thm 2.2] and [2, Thm. 6.1] (see also [16, 12.3 - 12.5] for the case in which x is the spectrum of a perfect field). In particular in the projective case we get for all S-points x

$$\mathrm{T}(\mathscr{C}H_0(X_1), \dots, \mathscr{C}H_0(X_n), \mathbb{G}_m^{\times r}) = \mathrm{K}(\kappa(x); \mathscr{C}H_0(X_1), \dots, \mathscr{C}H_0(X_n), \mathbb{G}_m^{\times r}),$$

where the K-group on the right is the Somekawa-type K-group defined by Raskind-Spieß and Akhtar in [26, Def. 2.1.1] and [2, 3.1].

(2) Identifying $\mathbb{G}_m$ with $\mathscr{C}H_0(S,1)$ one easily checks that the product structure on higher Chow groups induces a multi-linear map of reciprocity functors (in the sense of Definition 4.2.1, notice that the condition (L3) is automatic since we are in $\mathbf{RF}_1$)

$$\mathscr{C}H_0(X_1) \times \dots \times \mathscr{C}H_0(X_n) \times \mathbb{G}_m^{\times r} \to \mathscr{C}H_0(X_1 \times \dots \times X_n, r).$$

Hence we get a map from T(LHS) to the RHS and one can prove by hand that this is an isomorphism in a similar way as in [26] and [2].

5.3. **Relation with Milnor K-theory.** In the following we denote by $\mathbb{G}_m$ and $\mathrm{K}_n^{\mathrm{M}}$ the $\mathbb{Z}$-reciprocity functors over S defined in Example 2.4.6.

5.3.1. **Proposition.** *Let $\mathscr{M}$ be a $\mathbb{Z}$-reciprocity functor over S and x an S-point. For all $n \geqslant 1$ there exist a homomorphism of Abelian groups*

$$\mathscr{M}(x) \otimes_{\mathbb{Z}} \mathrm{K}_n^{\mathrm{M}}(x) \to \mathrm{T}(\mathscr{M}, \mathbb{G}_m^{\times n})(x), \quad m \otimes \{a_1, \dots, a_n\} \mapsto \tau(m, a_1, \dots, a_n).$$

Proof. The proof is similar to the proof of [22, Prop. 5.9]. Write $x = \operatorname{Spec} k$. Clearly there is a natural morphism of Abelian groups

$$\mathscr{M}(x) \otimes (k^\times)^{\otimes_{\mathbb{Z}} n} \to \mathrm{T}(\mathscr{M}, \mathbb{G}_m^{\times n})(x), \quad m \otimes a_1 \otimes \dots \otimes a_n \mapsto \tau(m, a_1, \dots, a_n).$$

Hence (using Corollary 4.2.5, (1)) it suffices to show that

$$\tau(m, a, 1-a, \underline{b}) = 0 \quad \text{in } \mathrm{T}(\mathscr{M}, \mathbb{G}_m^{\times n})(x), \tag{5.14}$$

for all $a, b_i \in k \setminus \{0, 1\}$, $m \in \mathscr{M}(x)$ and $\underline{b} = (b_2, \dots, b_n)$. For this take $c \in \bar{k}$ with $c^3 = a$, set $L = k(c)$ and $y = \operatorname{Spec} L$. Further, let $\mu \in \bar{k}$ be a third primitive root of 1, set $E = L(\mu)$ and $z = \operatorname{Spec} E$. Denote by $\varphi : z \to x$ the induced map. Notice that $[E : k]$ divides 6. Define a rational function f on $\mathbb{P}^1_x \supset \mathbb{A}^1_x = \operatorname{Spec} k[t]$

$$f := \frac{t^3 - a}{t^3 - (1+a)t^2 + (1+a)t - a} \in k(t).$$

Then in $E(t)$ we have

$$f = \frac{(t-c)(t-\mu c)(t-\mu^2 c)}{(t-a)(t+\mu)(t+\mu^2)}.$$

Further, let $\pi : \mathbb{P}^1_z \to z$ be the structure map, then by Remark 4.2.2

$$\tau(\pi^*\varphi^* m, t, 1-t, \pi^*\varphi^*\underline{b}) \in \mathrm{T}(\mathscr{M}, \mathbb{G}_m^{\times n})(\mathbb{P}^1_z, \{0\} + \{1\} + \{\infty\}).$$

Since $f \equiv 1 \bmod \{0\} + \{1\} + \{\infty\}$ reciprocity yields in $\mathrm{T}(\mathscr{M}, \mathbb{G}_m^{\times n})(z)$

$$\begin{aligned} 0 &= \sum_{P \in \mathbb{P}^1_z} (\tau(\pi^*\varphi^* m, t, 1-t, \pi^*\varphi^*\underline{b}), f)_P \\ &= \tau(\varphi^* m, c, 1-c, \varphi^*\underline{b}) + \tau(\varphi^* m, \mu c, 1-\mu c, \varphi^*\underline{b}) + \tau(\varphi^* m, \mu^2 c, 1-\mu^2 c, \varphi^*\underline{b}) \\ &\quad - \tau(\varphi^* m, c^3, 1-c^3, \varphi^*\underline{b}) - \tau(\varphi^* m, -\mu, 1+\mu, \varphi^*\underline{b}) - \tau(\varphi^* m, -\mu^2, 1+\mu^2, \varphi^*\underline{b}). \end{aligned}$$

Multiplying this by 3 we obtain in $\mathrm{T}(\mathscr{M}, \mathbb{G}_m^{\times n})(z)$

$$\begin{aligned} 0 = \tau(\varphi^* m, c^3, (1-c)(1-\mu c)(1-\mu^2 c), \varphi^*\underline{b}) - 3 \cdot \tau(\varphi^* m, c^3, 1-c^3, \varphi^*\underline{b}) \\ - \tau(\varphi^* m, -1, (1+\mu)(1+\mu^2), \varphi^*\underline{b}). \end{aligned}$$

Since $(1-c)(1-\mu c)(1-\mu^2 c) = 1 - c^3$ and $(1+\mu)(1+\mu^2) = 1$ we obtain

$$0 = 2 \cdot \tau(\varphi^* m, c^3, 1-c^3, \varphi^*\underline{b}) = 2 \cdot \varphi^*\tau(m, a, 1-a, \underline{b}) \quad \text{in } \mathrm{T}(\mathscr{M}, \mathbb{G}_m^{\times n})(z).$$

Applying φ_* we get

$$12 \cdot \tau(m, a, 1-a, \underline{b}) = 0 \quad \text{in } \mathrm{T}(\mathscr{M}, \mathbb{G}_m^{\times n})(x).$$

This holds for all S-points $x = \operatorname{Spec} k$ and all $a, b_i \in k \setminus \{0, 1\}$, $m \in M(x)$. Thus the vanishing (5.14) follows by exactly the same argument as in the proof of [22, Lemma 5.8]. (Notice that there is a misprint and the first $1-x$ in the displayed formula on page 32, line 13 should be replaced by $1-y$.) $\square$

5.3.2. **Proposition.** *The maps from Proposition 5.3.1 (with $\mathscr{M} = \mathbb{Z}$) for x running through all S-points yield a morphism of Mackey functors*

$$\sigma : \mathrm{K}_n^{\mathrm{M}} \to \mathrm{T}(\mathbb{G}_m^{\times n}) \quad \textit{in } \mathbf{MF}.$$

In particular $\sigma(x)$ is a surjective homomorphism of $\mathbb{Z}$-modules for all S-points x.

Proof. The compatibility with pullback is clear. It remains to show that for a finite morphism $\varphi : y = \operatorname{Spec} L \to x = \operatorname{Spec} k$ the pushforward φ_* is compatible with σ. This is really the same argument as in [25, Lemma 4.7]: Since both sides satisfy (MF1), (MF2) (see Remark 1.3.3), the arguments from [3, I, (5.9)] reduce us to the case in which every finite extension of k has degree a power of a fixed prime ℓ and hence can be written as a successive sequence of extensions of degree ℓ. Thus by functoriality we can assume $[L : k] = \ell$; hence $\mathrm{K}^{\mathrm{M}}_{n+1}(L) = \mathrm{K}^{\mathrm{M}}_n(k) \cdot \mathrm{K}^{\mathrm{M}}_1(L)$ (by [3, I, Cor. 5.3]) and the statement follows from the projection formula on both sides. The surjectivity statement follows immediately, since by definition (see Definition 4.2.3 and Theorem 4.2.4) any element in $\mathrm{T}(\mathbb{G}_m^{\times n})(x)$ is a finite sum of elements of the form $\mathrm{Tr}_{y/x}(\tau(a_1, \ldots, a_n))$ for y/x finite and $a_i \in \mathbb{G}_m(y)$. □

5.3.3. **Theorem.** *For all $n \geqslant 1$ the natural map*

$$\Phi : \mathbb{G}_m^{\times n} \to \mathrm{K}^{\mathrm{M}}_n, \quad (a_1, \ldots, a_n) \mapsto \{a_1, \ldots, a_n\}$$

is an n-linear map of reciprocity functors (in the sense of Definition 4.2.1) and the induced morphism

$$\mathrm{T}(\mathbb{G}_m^{\times n}) \xrightarrow{\simeq} \mathrm{K}^{\mathrm{M}}_n \quad \textit{in } \mathbf{MF}$$

is an isomorphism of Mackey functors *whose inverse is given by the map σ from Proposition 5.3.2 above. Moreover if the base field F is* infinite*, then the above is an isomorphism of* reciprocity functors*.*

Proof. Clearly Φ is an n-linear morphism of Mackey functors (Definition 4.1.5). Since $\mathrm{K}^{\mathrm{M}}_n \in \mathbf{RF}_1$ the condition (L3) in Definition 4.2.1 is automatically satisfied. Thus we obtain a morphism $\psi : \mathrm{T}(\mathbb{G}_m^{\times n}) \to \mathrm{K}^{\mathrm{M}}_n$ in $\mathbf{RF}$. Now σ is surjective and obviously we have $\psi \circ \sigma = \mathrm{id}$, hence it is an isomorphism in $\mathbf{MF}$. Notice that by (Inj) ψ is an isomorphism in $\mathbf{RF}$ if it is surjective on Nisnevich stalks. But by [6, Prop. 4.3] (see also [8]) $\mathrm{K}^{\mathrm{M}}_n(\mathscr{O}_{C,P}) \to \mathscr{K}_{n,C,P}$ is surjective if F is infinite, where $\mathscr{K}_{n,C,P}$ is defined in Example 2.4.6. This gives the second statement. □

5.3.4. *Remark.* (1) In the same way one can prove

$$\mathrm{K}^{\mathrm{M}}_n \cong \mathrm{T}(\mathrm{K}^{\mathrm{M}}_{n_1}, \ldots, \mathrm{K}^{\mathrm{M}}_{n_r}) \quad \text{for all } n = n_1 + \ldots + n_r,\ r \geqslant 1.$$

(2) Combining Theorem 5.3.3 and Corollary 5.2.5 we get the Nesterenko-Suslin isomorphism [25], i.e.

$$\mathrm{K}^{\mathrm{M}}_n(x) \cong \mathrm{CH}^n(x, n) \quad \text{for all } S\text{-points } x.$$

5.4. **Relation with Kähler differentials.** In the following we denote simply by $\mathbb{G}_a$, $\mathbb{G}_m$ and Ω^n the $\mathbb{Z}$-reciprocity functors over S defined in section 2. Recall $S = \operatorname{Spec} F$ with F a perfect field.

5.4.1. **Lemma.** *Let $\mathscr{M}_i$, $i = 1, \ldots, n$, be $\mathbb{Z}$-reciprocity functors. Then for any S-point $x = \operatorname{Spec} k$ the natural F-vector space structure on $k \otimes_{\mathbb{Z}} \mathscr{M}_1(x) \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} \mathscr{M}_n(x)$ extends naturally to $\mathrm{T}(\mathbb{G}_a, \mathscr{M}_1, \ldots, \mathscr{M}_n)(x)$.*

Proof. An element $\lambda \in F$ induces a morphism $\lambda : \mathbb{G}_a \to \mathbb{G}_a$ of reciprocity functors over S. We obtain a morphism $\lambda \otimes \mathrm{id} : \mathrm{T}(\mathbb{G}_a, \mathscr{M}_1, \ldots, \mathscr{M}_n) \to \mathrm{T}(\mathbb{G}_a, \mathscr{M}_1, \ldots, \mathscr{M}_n)$ of reciprocity functors over S and it is straightforward to check that this induces the looked for module structure $\mathrm{T}(\mathbb{G}_a, \mathscr{M}_1, \ldots, \mathscr{M}_n)(x)$. □

5.4.2. *Remark.* It is not clear that in the situation above, the natural k-vector space structure on the $\mathbb{Z}$-module $k \otimes_{\mathbb{Z}} \mathscr{M}_1(x) \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} \mathscr{M}_n(x)$ extends to the $\mathbb{Z}$-module $\mathrm{T}(\mathbb{G}_a, \mathscr{M}_1, \ldots, \mathscr{M}_n)(x)$.

5.4.3. **Lemma.** *Let $x = \operatorname{Spec} k$ be an S-point and let 0 be the zero point in the affine line $\mathbb{A}^1 = \operatorname{Spec} k[t] \subset \mathbb{P}^1$. Then for any $f \in k(t)^\times$ and $a, b_1, \dots, b_{n-1} \in k^\times$ we have*

$$(\tau(at, t, b_1, \dots, b_{n-1}), f)_0 = 0 \quad \textit{in } \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x).$$

Proof. We have

$$at \in \mathbb{G}_a(\mathbb{P}^1, 2 \cdot \{\infty\}), \quad t \in \mathbb{G}_m(\mathbb{P}^1, \{0\} + \{\infty\})$$

and

$$\underline{b} = (b_1, \dots, b_{n-1}) \in \mathbb{G}_m^{\times n}(\mathbb{P}^1).$$

Hence $\tau(at, t, \underline{b}) \in \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(\mathbb{P}^1, \{0\} + 2 \cdot \{\infty\})$ and

$$\begin{aligned}(\tau(at, t, \underline{b}), f)_0 &= (\tau(at, t, \underline{b}), f_0)_0 + (\tau(at, t, \underline{b}), f_0)_\infty \\ &= -\sum_{P \in \mathbb{P}^1 \setminus \{0, \infty\}} v_P(f_0) \mathrm{Tr}_{P/x} s_P(\tau(at, t, \underline{b})),\end{aligned}$$

for all $f_0 \in k(t)^\times$ with

$$f/f_0 \in U_0^{(1)} \text{ and } f_0 \in U_\infty^{(2)}. \tag{5.15}$$

We distinguish 3 cases. (In the calculation we use Lemma 5.4.1 without mentioning it.)

1. case: $f \in \mathscr{O}^\times_{\mathbb{A}^1, 0}$. In this case $f_0 := \frac{t^2 - f(0)}{t^2 - 1}$ satisfies (5.15). Let $\alpha \in \bar{k}$ be a root of $t^2 - f(0)$ and $\varphi : y = \operatorname{Spec} k(\alpha) \to x$ the induced map. First assume $\mathrm{char}(k) \neq 2$. Then we have in $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(y)$

$$\begin{aligned}\varphi^*(\tau(at, t, \underline{b}), f)_0 &= -\tau(a\alpha, \alpha, \underline{b}) - \tau(-a\alpha, -\alpha, \underline{b}) \\ &\quad + \tau(a, 1, \underline{b}) + \tau(-a, -1, \underline{b}) \\ &= \tau(-a\alpha, \alpha, \underline{b}) + \tau(-a\alpha, \frac{-1}{\alpha}, \underline{b}) \\ &= \tau(-a\alpha, -1, \underline{b}) = 0.\end{aligned}$$

Since φ has degree 1 or 2, property (MF2) implies that $(\tau(at, t, \underline{b}), f)_0$ is zero in $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x)$. If $\mathrm{char}(k) = 2$ we obtain

$$(\tau(at, t, \underline{b}), f)_0 = -2 \cdot \mathrm{Tr}_{y/x}(\tau(a\alpha, \alpha, \underline{b})) + 2 \cdot \tau(a, 1, \underline{b}) = 0.$$

2. case: $f = t$. In this case $f_0 := \frac{t(t+1)}{t^2 + t + 1}$ satisfies (5.15). Let $\alpha \in \bar{k}$ be a third primitive root of 1 and $y = \operatorname{Spec} k(\alpha) \to x$ the induced map. Assume first $\mathrm{char}(k) \neq 3$ and y has degree 2 over x. Then we have in $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x)$

$$(\tau(at, t, \underline{b}), f)_0 = -\tau(-a, -1, \underline{b}) + \mathrm{Tr}_{y/x}(\tau(a\alpha, \alpha, \underline{b})) = \mathrm{Tr}_{y/x}(\tfrac{1}{3}\tau(a\alpha, 1, \underline{b})) = 0.$$

If the degree of y over x is 1, then

$$(\tau(at, t, \underline{b}), f)_0 = -\tau(-a, -1, \underline{b}) + \tau(a\alpha, \alpha, \underline{b}) + \tau(a\alpha^2, \alpha^2, \underline{b}) = 0.$$

If $\mathrm{char}(k) = 3$, then

$$(\tau(at, t, \underline{b}), f)_0 = -\tau(-a, -1, \underline{b}) + 2 \cdot \tau(a, 1, \underline{b}) = 0.$$

3. case: $f \in k(t)^\times$ *arbitrary.* Write $f = t^n u$ with $u \in \mathscr{O}^\times_{\mathbb{A}^1, 0}$ and conclude with the first two cases. □

5.4.4. **Proposition.** *Let $x = \operatorname{Spec} k$ be an S -point. For all $n \geqslant 1$ there exists a homomorphism of F-vector spaces (see Lemma 5.4.1)*

$$\theta(x) : \Omega^n_x \to \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x), \quad a\frac{db_1}{b_1} \cdots \frac{db_n}{b_n} \mapsto \tau(a, b_1, \dots, b_n).$$

Proof. By [5, Lemma (4.2)] (straightforward to check!) the kernel of the surjective map

$$k \otimes_{\mathbb{Z}} (k^\times)^{\otimes n} \to \Omega^n_x, \quad a \otimes (b_1 \otimes \ldots \otimes b_n) \mapsto a\frac{db_1}{b_1} \cdots \frac{db_n}{b_n}$$

is the subgroup of $k \otimes_{\mathbb{Z}} (k^\times)^{\otimes n}$ generated by elements of the following form

$$a \otimes b_1 \otimes \ldots \otimes b_n, \quad \text{with } b_i = b_j \text{ for some } 1 \leqslant i < j \leqslant n, \tag{5.16}$$

and

$$\sum_{i=1}^{r} \lambda a_i \otimes a_i \otimes b_1 \otimes \ldots \otimes b_{n-1} - \sum_{i=1}^{s} \lambda a'_i \otimes a'_i \otimes b_1 \otimes \ldots \otimes b_{n-1}, \tag{5.17}$$

where $\lambda, a_i, a'_i \in k^\times$ with $\sum_{i=1}^r a_i = \sum_{i=1}^s a'_i$. Thus it suffices to check, that the natural map

$$k \otimes_{\mathbb{Z}} (k^\times)^{\otimes n} \to \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x), \quad a \otimes b_1 \otimes \ldots \otimes b_n \mapsto \tau(a, b_1, \ldots, b_n)$$

maps the elements (5.16) and (5.17) to 0. By Proposition 5.3.1 (with $\mathscr{M} = \mathbb{G}_a$) this map factors over $k \otimes_{\mathbb{Z}} \mathrm{K}^{\mathrm{M}}_n(k)$, which shows (using also Lemma 5.4.1) that (5.16) is mapped to zero. Now take $\lambda, a_i, a'_i, b_i \in k^\times$ as in (5.17) and write 0 for the zero point in the affine line $\mathbb{A}^1 = \operatorname{Spec} k[t] \subset \mathbb{P}^1$. We get

$$\tau(\lambda t, t, \underline{b}) \in \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(\mathbb{P}^1, 0 + 2 \cdot \infty). \tag{5.18}$$

Set

$$f := \frac{t^s \prod_{i=1}^r (t - a_i)}{t^r \prod_{i=1}^s (t - a'_i)}.$$

Since $\sum_{i=1}^r a_i = \sum_{i=1}^s a'_i$ we have $f \equiv 1 \bmod 2 \cdot \infty$. Thus $(\tau(\lambda t, t, \underline{b}), f)_\infty = 0$. Further $(\tau(\lambda t, t, \underline{b}), f)_0 = 0$ by Lemma 5.4.3. Hence

$$\begin{aligned} 0 = \sum_{P \in \mathbb{P}^1} (\tau(\lambda t, t, \underline{b}), f)_P &= \sum_{P \in \mathbb{P}^1 \setminus \{0, \infty\}} (\tau(\lambda t, t, \underline{b}), f)_P \\ &= \sum_{i=1}^r \tau(\lambda a_i, a_i, \underline{b}) - \sum_{i=1}^s \tau(\lambda a'_i, a'_i, \underline{b}). \end{aligned}$$

This yields the statement. □

5.4.5. **Corollary.** *Let X be in* $\mathrm{RegCon}^{\leqslant 1}$ *with generic point η. Then the morphism $\theta(\eta)$ from above induces a map*

$$\theta(X) : \Omega^n(X) \to \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(X).$$

Proof. Since any reciprocity functor $\mathscr{M}$ is in particular a Zariski sheaf on X and we have inclusions $\mathscr{M}(X) \subset \mathscr{M}(\eta)$, it suffices to check that $\theta(\eta)$ sends $\Omega^n_{X,P}$ to $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})_{X,P}$ for all closed points $P \in X$. But this follows immediately from the fact that any element in $\Omega^n_{X,P}$ is a sum of elements of the form $a\frac{db_1}{b_1} \cdots \frac{db_n}{b_n}$ with $a \in \mathscr{O}_{X,P}$ and $b_i \in \mathscr{O}^\times_{X,P}$. □

5.4.6. **Proposition.** *The maps $\theta(X)$ from Corollary 5.4.5 for X running through* $\mathrm{RegCon}^{\leqslant 1}$ *yield a morphism of reciprocity functors*

$$\theta : \Omega^n \to \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n}) \quad \textit{in } \mathbf{RF}.$$

In particular θ is a surjective homomorphism of Nisnevich sheaves.

Proof. The compatibility of θ with pullbacks is immediate. It suffices to check the compatibility with pushforward on S-points and to consider the two cases in which $y = \operatorname{Spec} L \to x = \operatorname{Spec} k$ is either separable or purely inseparable of degree p. In

the separable case, we can write any element in $\Omega^n_L = L \otimes \Omega^n_k$ as a finite sum of elements of the form

$$a\frac{db_1}{b_1}\cdots\frac{db_n}{b_n}, \quad \text{with } a \in L \text{ and } b_i \in k.$$

Thus the compatibility with the trace follows since both sides satisfy the projection formula. In the purely inseparable case we write $L = k[c]$ with $c^p = a \in k$, where $p = \mathrm{char}(k) > 0$. We can write any element in Ω^n_L as a sum of elements of the following form

$$\lambda c^j\frac{db_1}{b_1}\cdots\frac{db_n}{b_n}, \quad \text{with } 0 \leqslant j \leqslant p-1,\ \lambda, b_i \in k \tag{5.19}$$

and

$$\lambda c^j\frac{dc}{c}\frac{db_1}{b_1}\cdots\frac{db_{n-1}}{b_{n-1}}, \quad \text{with } 1 \leqslant j \leqslant p,\ \lambda, b_i \in k. \tag{5.20}$$

The compatibility with the trace for the elements (5.19) again follows from the projection formula (in fact the trace is zero in this case). The compatibility for the elements (5.20) in the case $j = p$ follows from the projection formula on both sides together with (Tr7) in 2.5.1 (since $c^p \in k$). Further, by [20, Ex. 16.6] we have

$$\mathrm{Tr}_{L/k}(c^j\frac{dc}{c}\frac{b_1}{b_1}\cdots\frac{db_{n-1}}{b_{n-1}}) = 0, \quad \text{for all } 1 \leqslant j \leqslant p-1.$$

Thus it remains to show that $\mathrm{Tr}_{L/k}(\tau(\lambda c^j, c, \underline{b})) = 0$ in $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x)$ for $1 \leqslant j \leqslant p-1$. For this we view $y = V(t^p - a) \subset \mathbb{A}^1 = \operatorname{Spec} k[t] \subset \mathbb{P}^1$ as a closed point. Then for $1 \leqslant j \leqslant p-1$ reciprocity yields

$$\begin{aligned}
\mathrm{Tr}_{L/k}(\tau(\lambda c^j, c, \underline{b})) &= -(\tau(\lambda t^j, t, \underline{b}), t^p - a)_0 - (\tau(\lambda t^j, t, \underline{b}), t^p - a)_\infty & (5.21)\\
&= -\tfrac{1}{j}(\tau(\lambda t^j, t^j, \underline{b}), t^p - a)_0 - \tfrac{1}{j}(\tau(\lambda t^j, t^j, \underline{b}), t^p - a)_\infty. & (5.22)
\end{aligned}$$

Now let $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ be the x-morphism induced by $k[t] \to k[t]$, $t \mapsto t^j$. Then by Proposition 1.5.5, 2. and Lemma 5.4.3 we have

$$(\tau(\lambda t^j, t^j, \underline{b}), t^p - a)_0 = (\tau(\lambda t, t, \underline{b}), \pi_*(t^p - a))_0 = 0.$$

For the same reason also

$$(\tau(\lambda t^j, t^j, \underline{b}), t^p)_0 = (\tau(\lambda t, t, \underline{b}), \pi_*(t^p))_0 = 0.$$

Hence by reciprocity also

$$(\tau(\lambda t^j, t^j, \underline{b}), t^p)_\infty = 0.$$

We obtain

$$\mathrm{Tr}_{L/k}(\tau(\lambda c^j, c, \underline{b})) = -\tfrac{1}{j}(\tau(\lambda t^j, t^j, \underline{b}), \frac{t^p - a}{t^p})_\infty$$

which is zero since $\tau(\lambda t^j, t^j, \underline{b}) \in \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(\mathbb{P}^1, 0 + (j+1)\cdot\{\infty\})$ and $\frac{t^p-a}{t^p} \in U^{(p)}_\infty$.

Hence θ is a morphism of reciprocity functors. Finally, for $X \in \mathrm{RegCon}^{\leqslant 1}$ and $P \in X$ closed any element in $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})^h_{X,P}$ (the Nisnevich stalk on X in P) is a finite sum of elements of the form $\mathrm{Tr}_{Y/X}(\tau(a, b_1, \ldots, b_n))$, with $\pi : Y \to X$ in $\mathrm{RegCon}^{\leqslant 1}_*$, $a \in \pi_*(\mathscr{O}_Y)^h_P$ and $b_i \in \pi_*(\mathscr{O}_Y^\times)^h_P$ by definition (see Definition 4.2.3 and Theorem 4.2.4). Thus the surjectivity statement follows immediately. $\square$

5.4.7. **Theorem.** *Assume F has characteristic zero. Then the map*

$$\theta : \Omega^n \xrightarrow{\simeq} \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})$$

from Proposition 5.4.6 is an isomorphism of reciprocity functors.

Before we prove the theorem we need the following Lemma.

5.4.8. **Lemma.** *Assume F has characteristic zero. Then for all $C \in (\mathscr{C}/S)$, $P \in C$ and $r \geqslant 1$*

$$\mathrm{Fil}^r_P \mathbb{G}_a(\eta_C) = \{a \in \kappa(\eta_C) \,|\, v_P(a) \geqslant -r+1\}.$$

In particular

$$\mathrm{Fil}^0_P \mathbb{G}_a(\eta_C) = \mathrm{Fil}^1_P \mathbb{G}_a(\eta_C) = \mathscr{O}_{C,P}.$$

Proof. Recall that the local symbol of $\mathbb{G}_a$ at a point $P \in C$ is given by (see Theorem 2.5.5)

$$(a, f)_P = \mathrm{Res}_P(a \frac{df}{f}).$$

The inclusion $\supset$ is thus straightforward to check. For the other inclusion take $a \in \mathrm{Fil}^r_P \mathbb{G}_a(\eta_C)$ and assume we can write $a = u/t^s$ with $s \geqslant r$, $u \in \mathscr{O}^\times_{C,P}$ and $t \in \mathscr{O}_{C,P}$ a local parameter. Then for $b \in \mathscr{O}_{C,P}$ we have by definition of Fil

$$\begin{aligned} 0 = (a, 1+bt^s)_P &= \mathrm{Res}_P \left(\frac{u}{t^s} \frac{d(1+bt^s)}{1+bt^s} \right) \\ &= \mathrm{Res}_P \left(\frac{sub}{1+bt^s} \frac{dt}{t} \right) + \mathrm{Res}_P \left(\frac{ub}{1+bt^s} db \right) = s\mathrm{Tr}_{P/x_c}(u(P)b(P)). \end{aligned}$$

Since $\mathrm{Tr} : \kappa(P) \times \kappa(P) \to \kappa(x_C)$, $(\lambda, \mu) \mapsto \mathrm{Tr}(\lambda\mu)$ is non-degenerate, we get $u(P) = 0$. A contradiction. $\square$

Proof of Theorem 5.4.7. For $X \in \mathrm{RegCon}^{\leqslant 1}$ define

$$\Phi_X : \mathbb{G}_a(X) \times (\mathbb{G}_m(X))^{\times n} \to \Omega^n(X), \quad (a, b_1, \ldots, b_n) \mapsto a \frac{db_1}{b_1} \cdots \frac{db_n}{b_n}.$$

Clearly the collection $\Phi = \{\Phi_X\}_{X \in \mathrm{RegCon}^{\leqslant 1}}$ satisfies condition (L1) of Definition 4.1.5 and by (Tr1), (Tr2) and (Tr7) in 2.5.1 also condition (L2). Now let $C \in (\mathscr{C}/S)$ be a curve, $P \in C$ a point and $r_0, \ldots, r_n \geqslant 1$ positive integers. Take $a \in \mathrm{Fil}^{r_0}_P \mathbb{G}_a(\eta_C)$ and $b_i \in \mathrm{Fil}^{r_i}_P \mathbb{G}_m(\eta_C) = \mathbb{G}_m(\eta_C)$, $i = 1, \ldots, n$. Then we can write $a = \frac{a_0}{t^{r_0-1}}$ with $a_0 \in \mathscr{O}_{C,P}$ (by Lemma 5.4.8) and $b_i = t^{s_i} u_i$ with $s_i \in \mathbb{Z}$ and $u_i \in \mathscr{O}^\times_{C,P}$. Then for $u = 1 + t^r c \in U^{(r)}_P$ with $r := \max\{r_0, \ldots, r_n\}$ we get

$$\begin{aligned} \mathrm{Res}_P(\Phi(a, b_1 \ldots, b_n) \frac{du}{u}) &= \mathrm{Res}_P \left(\frac{a_0}{t^{r_0-1}} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n} \frac{t^{r-1}(rcdt + tdc)}{1+t^r c} \right) \\ &+ \sum_{i=1}^n s_i \mathrm{Res}_P \left(\frac{a_0}{t^{r_0-1}} \frac{du_1}{u_1} \cdots \underbrace{\frac{dt}{t}}_{i\text{-th place}} \cdots \frac{du_n}{u_n} \frac{t^{r-1}(rcdt + tdc)}{1+t^r c} \right), \end{aligned}$$

which clearly is zero. Hence $\Phi(a, b_1 \ldots, b_n) \in \mathrm{Fil}^r_P \Omega^n(\eta_C)$ and thus satisfies (L3). Therefore Φ is an (n+1)-linear map of reciprocity functors and the universal property of $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})$ yields a map of reciprocity functors (also denoted by Φ)

$$\Phi : \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n}) \to \Omega^n.$$

By the very definition of Φ and θ we have $\Phi \circ \theta = \mathrm{id}_{\Omega^n}$ (it suffices to check this on S-points). In particular θ is injective. Since it is also surjective by Proposition 5.4.6, the theorem follows. $\square$

5.4.9. *Remark.* The above theorem should be compared to [11, Thm. 3.6]. In particular

$$\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n}) = \mathrm{K}(F; \mathbb{G}_a, \mathbb{G}_m^{\times n}),$$

where the K-group on the right is the Somekawa-type K-group defined by Hiranouchi in [11, Def. 3.3].

5.4.10. Assume F has characteristic $p > 0$. Notice that $d : \Omega^{n-1} \to \Omega^n$ is a map of $\mathbb{Z}$-reciprocity functors and thus

$$X \mapsto B_1\Omega^n(X) := (d\Omega_X^{n-1})(X)$$

defines a Nisnevich sheaf with transfers on $\mathrm{Reg}^{\leqslant 1}$, $B_1\Omega^n \in \mathbf{NT}$. We denote by $\Omega^n/B_1\Omega^n \in \mathbf{NT}$ the quotient in $\mathbf{NT}$. Furthermore for any $X \in \mathrm{Reg}^{\leqslant 1}$ we have the inverse Cartier operator

$$C^{-1} : \Omega^n_X \to \Omega^n_X/B_1\Omega^n_X,$$

which on stalks is given by

$$C^{-1} : \Omega^n_{X,P} \to \Omega^n_{X,P}/B_1\Omega^n_{X,P}, \quad a\frac{db_1}{b_1}\cdots\frac{db_n}{b_n} \mapsto a^p\frac{db_1}{b_1}\cdots\frac{db_n}{b_n}.$$

The Cartier operator is clearly compatible with pullbacks and it is well-known that it is also compatible with the trace (see e.g. [28, Thm. 2.6, (i), (v)] and use that the Frobenius on the de Rham-Witt complex lifts the inverse Cartier operator). Thus we get a morphism of Nisnevich sheaves with transfers on $\mathrm{Reg}^{\leqslant 1}$

$$C^{-1} : \Omega^n \to \Omega^n/B_1\Omega^n \quad \text{in } \mathbf{NT}.$$

Recursively we define $B_r\Omega^n \in \mathbf{NT}$ for $r \geqslant 2$ as the Nisnevich sheaf with transfers which as a Zariski sheaf on $X \in \mathrm{Reg}^{\leqslant 1}$ is defined as the preimage of $C^{-1}(B_{r-1}\Omega^n_X)$ under the natural surjection $\Omega^n_X \to \Omega^n_X/B_1$ (see e.g. [14, (2.2)]). We obtain a chain in $\mathbf{NT}$

$$0 := B_0 \subset B_1 \subset \ldots \subset B_r \subset \ldots \subset \Omega^n$$

and we set

$$B_\infty\Omega^n := \bigcup_r B_r\Omega^n \subset \Omega^n \quad \text{in } \mathbf{NT}.$$

The the inverse Cartier operator induces thus an endomorphism in $\mathbf{NT}$

$$C^{-1} : \Omega^n/B_\infty \to \Omega^n/B_\infty.$$

5.4.11. *Remark.* Clearly the $B_i\Omega^n$'s defined above are also reciprocity functors. But we don't know whether B_1 is the image of d in $\mathbf{RF}$ or whether Ω^n/B_r, $r \leqslant \infty$, is a reciprocity functor (condition (Inj) is not clear). But Ω^n/B_r is a lax Mackey functor with specialization map and hence we get a reciprocity functor $\Sigma(\Omega^n/B_r)$. In view of the corollary below it would be nice to know whether for an S-point x the natural surjection $\Omega^n_x/B_r(x) \to \Sigma(\Omega^n/B_r)(x)$ is actually an isomorphism; for this we need to check (see Lemma 3.2.5) whether $s_P(\Omega^n_{C,P} \cap B_r(\eta_C))$ is contained in $B_r(P)$, for all $C \in (\mathscr{C}/S)$ and all $P \in C$. In case P is separable over x_C this follows easily from the existence of a fine residue map (c.f. 2.5.3) $\widetilde{\mathrm{Res}}_P : \Omega^n_{\eta_C} \to \Omega^{n-1}_P$, which satisfies $\widetilde{\mathrm{Res}}_P(a\frac{dt}{t}) = s_P(a)$, for $a \in \Omega^n_{C,P}$ and t a local parameter at t, and which sends $B_r\Omega^n_{\eta_C} \cdot \frac{dt}{t}$ to $B_r\Omega^n_P$. But in general we don't know whether it is true.

5.4.12. **Corollary.** *Assume F has characteristic $p > 0$. Then the surjective morphism of reciprocity functors $\theta : \Omega^n \to \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^\times)$ from Proposition 5.4.4 factors via*

$$\Omega^n/B_\infty \twoheadrightarrow \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^\times) \quad \textit{in } \mathbf{NT}$$

and the following diagram commutes (in $\mathbf{NT}$)

$$\begin{tikzcd}
\Omega^n/B_\infty \arrow[r, "C^{-1}"] \arrow[d, two heads] & \Omega^n/B_\infty \arrow[d, two heads] \\
\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n}) \arrow[r, "F\otimes\mathrm{id}"] & \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^\times),
\end{tikzcd}$$

where $F : \mathbb{G}_a \to \mathbb{G}_a$ is the absolute Frobenius.

Proof. The commutativity of the diagram is immediate once we know the vertical maps exist. Thus we have to show that for any $X \in \mathrm{RegCon}^{\leqslant 1}$ and any $r \geqslant 1$ the map θ sends $B_r(X)$ to zero. Since $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})$ is a reciprocity functor it suffices to prove this for all S-points x. First we show that $B_1(x)$ is mapped to zero in $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x)$, i.e. we have to show that for $a \in \kappa(x)^\times$ and $\underline{b} \in \mathbb{G}_m(x)^{\times n-1}$ we have

$$\tau(a, a, \underline{b}) = 0 \quad \text{in } \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x).$$

We can assume that $\kappa(x)$ contains a $(p-1)$-th primitive root of unity ζ, else we consider $\varphi : \operatorname{Spec} k(x)(\zeta) \to x$, which is of degree prime to p and get

$$\deg \varphi \cdot \tau(a, a, \underline{b}) = \varphi_* \tau(\varphi^* a, \varphi^* a, \varphi^* \underline{b}) = 0.$$

Further by Remark 4.2.10 we can also assume ζ lies in our ground field F. Proposition 5.4.4 implies

$$\tau(a, a, \underline{b}) = \tau(a+1, a+1, \underline{b}) \quad \text{in } \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x).$$

Applying $F \otimes \mathrm{id}$ we obtain

$$\tau(a^p, a, \underline{b}) = \tau((a+1)^p, a+1, \underline{b}).$$

On the other hand, Proposition 5.4.4 yields

$$\tau((a+1)^p, a+1, \underline{b}) = \tau((a+1)^{p-1} a, a, \underline{b}) \quad \text{in } \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x).$$

So that all in all we obtain

$$\tau\left((a^{p-1} - (a+1)^{p-1})a, a \otimes \underline{b}\right) = 0 \quad \text{in } \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x). \tag{5.23}$$

Consider $\mathbb{A}^1 = \operatorname{Spec} \kappa(x)[t] \subset \mathbb{P}^1$. Since $(t^{p-1} - (t+1)^{p-1})a \in \mathbb{G}_a(\mathbb{P}^1, (p-1) \cdot \{\infty\})$ we have

$$\tau((t^{p-1} - (t+1)^{p-1})a, a, \underline{b}) \in \mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(\mathbb{P}^1, (p-1) \cdot \{\infty\}).$$

Further $f := \frac{t^{p-1} - a^{p-1}}{t^{p-1}} \in \mathbb{G}_m(\eta_{\mathbb{P}^1})$ is congruent to 1 modulo $(p-1) \cdot \{\infty\}$. Thus reciprocity yields

$$\begin{aligned} 0 &= \sum_{P \in \mathbb{A}^1} v_P(f) \mathrm{Tr}_{P/x} s_P(\tau((t^{p-1} - (t+1)^{p-1})a, a, \underline{b})) \\ &= \sum_{i=0}^{p-1} \tau\left(((a\zeta^i)^{p-1} - (a\zeta^i + 1)^{p-1})a, a, \underline{b}\right) + (p-1)\tau(a, a, \underline{b}\right). \end{aligned}$$

Using $\zeta \in F$, Lemma 5.4.1, $\tau(c, \zeta^i, \underline{b}) = 0$ for all $c \in \mathbb{G}_a(x)$, all i, and (5.23) for $a\zeta^i$ we obtain

$$\begin{aligned} 0 &= \sum_{i=0}^{p-1} \zeta^{-i} \cdot \tau\left(((a\zeta^i)^{p-1} - (a\zeta^i + 1)^{p-1})\zeta^i a, \zeta^i a, \underline{b}\right) + (p-1)\tau(a, a, \underline{b}) \\ &= (p-1)\tau(a, a, \underline{b}). \end{aligned}$$

Hence θ maps B_1 to zero. By definition of B_n, the image of B_n in $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x)$ equals $F \otimes \mathrm{id}$(image of B_{n-1} in $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times n})(x)$) and hence vanishes by induction. This gives the statement. □

5.4.13. *Remark.* One would like to construct a map from $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_m^{\times} n)$ to $\Sigma(\Omega^n / B_\infty)$ using the universal property as in the proof of Theorem 5.4.7. The problem is that we don't have a description of $\mathrm{Fil}^r_P \mathbb{G}_a(\eta_C)$ in positive characteristic. In case P is separable over x_C this can be done (cf. [18, Prop.6.4]). But the points which are inseparable over x_C make trouble.

5.5. **A vanishing for unipotent groups.**

5.5.1. **Theorem.** *Assume* $\mathrm{char}(\kappa(S)) \neq 2$. *Let* $\mathscr{M}_1, \ldots, \mathscr{M}_n$ *be* $\mathbb{Z}$*-reciprocity functors and assume that at least* two *of them are smooth connected commutative* unipotent *group schemes over* S. *Then*

$$\mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n) = 0.$$

Proof. We may assume that $\mathscr{M}_1 = G$ and $\mathscr{M}_2 = H$ are smooth connected commutative unipotent group schemes over S. Then G and H have subnormal series whose quotients are isomorphic to $\mathbb{G}_a$ (see e.g. [23, XV, Prop. 2.20]). If $0 \subset G_r \subset \ldots \subset G_1 \subset G$ is such a series for G, then all the G_i's are again smooth, connected, commutative and unipotent (see e.g. the comment before [23, XV, Prop. 2.20]) and satisfy $H^1_{\mathrm{fppf}}(\mathscr{O}^h_{X,x}, G_i) = 0$ for all $X \in \mathrm{Reg}^{\leqslant 1}$, $x \in X$ (since $H^1_{\mathrm{fppf}}(\mathscr{O}^h_{X,x}, \mathbb{G}_a) = 0$). Thus the sequences $0 \to G_i \to G_{i-1} \to G_i/G_{i-1} \cong \mathbb{G}_a \to 0$ are exact also in **RF** (by Lemma 3.2.12). Since the same works for H the right exactness of T (see Corollary 4.2.9) reduces us to the case $G = H = \mathbb{G}_a$. In this case it suffices to show that for any S-point x and any elements $a, b \in \mathbb{G}_a(x)$, $m_i \in \mathscr{M}_i(x)$ we have (with $\underline{m} = (m_2, \ldots, m_n)$)

$$\tau(a, b, \underline{m}) = 0 \quad \text{in } \mathrm{T}(\mathbb{G}_a, \mathbb{G}_a, \mathscr{M}_2, \ldots, \mathscr{M}_n)(x). \tag{5.24}$$

To show this consider $\pi : \mathbb{P}^1_x \to x$ and write $\mathbb{P}^1_x \setminus \{\infty\} = \mathrm{Spec}\, \kappa(x)[t]$. Then $\pi^*(a) \cdot t, \pi^*(b) \cdot t \in \mathbb{G}_a(\mathbb{P}^1_x, 2\{\infty\})$ and $\pi^*(m_i) \in \mathscr{M}_i(\mathbb{P}^1)$. Thus

$$\tau_0 := \tau(\pi^*(a) \cdot t, \pi^*(b) \cdot t, \pi^*(\underline{m})) \in \mathrm{T}(\mathbb{G}_a, \mathbb{G}_a, \underline{\mathscr{M}})(\mathbb{P}^1_x, 2\{\infty\}).$$

Define $f := t^2/(t^2 - 1) \in \kappa(x)(t)$, which is congruent to 1 modulo $2\{\infty\}$. Then the reciprocity law yields

$$0 = \sum_{P \in \mathbb{P}^1_x} (\tau_0, f)_P = -\tau(a, b, \underline{m}) - \tau(-a, -b, \underline{m}) = -2\tau(a, b, \underline{m}),$$

which proves (5.24). □

References

1. *Théorie des topos et cohomologie étale des schémas. tome 3*, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.
2. Reza Akhtar, *Milnor K-theory of smooth varieties*, *K*-Theory. An Interdisciplinary Journal for the Development, Application, and Influence of *K*-Theory in the Mathematical Sciences **32** (2004), no. 3, 269–291.
3. H. Bass and J. Tate, *The Milnor ring of a global field*, Algebraic *K*-theory, II: "Classical" algebraic *K*-theory and connections with arithmetic (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972), Springer, Berlin, 1973, pp. 349–446. Lecture Notes in Math., Vol. 342.
4. Spencer Bloch and Hélène Esnault, *The additive dilogarithm*, Documenta Mathematica (2003), no. Extra Vol., 131–155 (electronic), Kazuya Kato's fiftieth birthday.
5. Spencer Bloch and Kazuya Kato, *p-adic étale cohomology*, Institut des Hautes Études Scientifiques. Publications Mathématiques (1986), no. 63, 107–152.
6. Philippe Elbaz-Vincent and Stefan Müller-Stach, *Milnor K-theory of rings, higher Chow groups and applications*, Inventiones Mathematicae **148** (2002), no. 1, 177–206.
7. William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998. MR 1644323 (99d:14003)
8. Ofer Gabber, *Letter to Bruno Kahn*, (1998).
9. Thomas Geisser and Marc Levine, *The K-theory of fields in characteristic p*, Inventiones Mathematicae **139** (2000), no. 3, 459–493.
10. A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Institut des Hautes Études Scientifiques. Publications Mathématiques (1965), no. 24, 231.

11. Toshiro Hiranouchi, *Somekawa's K-groups and additive higher Chow groups*, http://arxiv.org/abs/1208.6455 (2012).
12. Annette Huber and Bruno Kahn, *The slice filtration and mixed Tate motives*, Compos. Math. **142** (2006), no. 4, 907–936. MR 2249535 (2007e:14034)
13. Reinhold Hübl and Ernst Kunz, *On algebraic varieties over fields of prime characteristic*, Arch. Math. **62** (1994), no. 1, 88–96.
14. Luc Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Annales Scientifiques de l'École Normale Supérieure. Quatrième Série **12** (1979), no. 4, 501–661.
15. Bruno Kahn, *Nullité de certains groupes attachés aux variétés semi-abéliennes sur un corps fini; application*, Comptes Rendus de l'Académie des Sciences. Série I. Mathématique **314** (1992), no. 13, 1039–1042.
16. Bruno Kahn and Takao Yamazaki, *Somekawa's K-groups and Voevodsky's Hom groups*, http://arxiv.org/abs/1108.2764 (2011).
17. Kazuya Kato, *Milnor K-theory and the Chow group of zero cycles*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, p. 241–253.
18. Kazuya Kato and Henrik Russell, *Modulus of a rational map into a commutative algebraic group*, Kyoto Journal of Mathematics **50** (2010), no. 3, 607–622.
19. Moritz Kerz, *The Gersten conjecture for Milnor K-theory*, Inventiones Mathematicae **175** (2009), no. 1, 1–33.
20. Ernst Kunz, *Kähler differentials*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1986.
21. Marc Levine, *Mixed motives*, Mathematical Surveys and Monographs, vol. 57, American Mathematical Society, Providence, RI, 1998. MR 1623774 (99i:14025)
22. Carlo Mazza, Vladimir Voevodsky, and Charles Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI, 2006.
23. James S. Milne, *Basic Theory of Affine Group Schemes*, 2012, Available at www.jmilne.org/math/.
24. Fabien Morel, Lecture Notes in Mathematics, vol. 2052, Springer-Verlag, to appear.
25. Yu. P. Nesterenko and A. A. Suslin, *Homology of the general linear group over a local ring, and Milnor's K-theory*, Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya **53** (1989), no. 1, 121–146.
26. Wayne Raskind and Michael Spiess, *Milnor K-groups and zero-cycles on products of curves over p-adic fields*, Compositio Mathematica **121** (2000), no. 1, 1–33.
27. Markus Rost, *Chow groups with coefficients*, Documenta Mathematica **1** (1996), No. 16, 319–393 (electronic).
28. Kay Rülling, *The generalized de Rham-Witt complex over a field is a complex of zero-cycles*, Journal of Algebraic Geometry **16** (2007), no. 1, 109–169.
29. Jean-Pierre Schneiders, *Quasi-abelian categories and sheaves*, Mémoires de la Société Mathématique de France. Nouvelle Série (1999), no. 76, vi+134.
30. Jean-Pierre Serre, *Algèbre locale. Multiplicités*, Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel. Seconde édition, 1965. Lecture Notes in Mathematics, vol. 11, Springer-Verlag, Berlin, 1965.
31. ———, *Groupes algébriques et corps de classes*, second ed., Publications de l'Institut Mathématique de l'Université de Nancago, 7, Hermann, Paris, 1984, Actualités Scientifiques et Industrielles, 1264.
32. M. Somekawa, *On Milnor K-groups attached to semi-abelian varieties*, K-Theory. An Interdisciplinary Journal for the Development, Application, and Influence of K-Theory in the Mathematical Sciences **4** (1990), no. 2, 105–119.
33. Andrei Suslin and Vladimir Voevodsky, *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, p. 117–189.
34. Vladimir Voevodsky, *Cohomological theory of presheaves with transfers*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, p. 87–137.
35. ———, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, p. 188–238.
36. ———, *Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic*, Int. Math. Res. Not. (2002), no. 7, 351–355. MR 1883180 (2003c:14021)
37. ———, *Cancellation theorem*, Doc. Math. (2010), no. Extra volume: Andrei A. Suslin sixtieth birthday, 671–685. MR 2804268 (2012d:14035)

38. ———, *Unstable motivic homotopy categories in Nisnevich and cdh-topologies*, J. Pure Appl. Algebra **214** (2010), no. 8, 1399–1406. MR 2593671 (2011e:14041)

Florian Ivorra: Institut de recherche mathématique de Rennes, UMR 6625 du CNRS, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes cedex (France)
E-mail address: florian.ivorra@univ-rennes1.fr

Kay Rülling: FB Mathematik, Universität Duisburg-Essen, Campus Essen, 45117 Essen (Germany)
E-mail address: kay.ruelling@uni-due.de